First Edition

Pascal

Newton

Understanding the Principles of Mathematics

Pythagoras

Evaristo
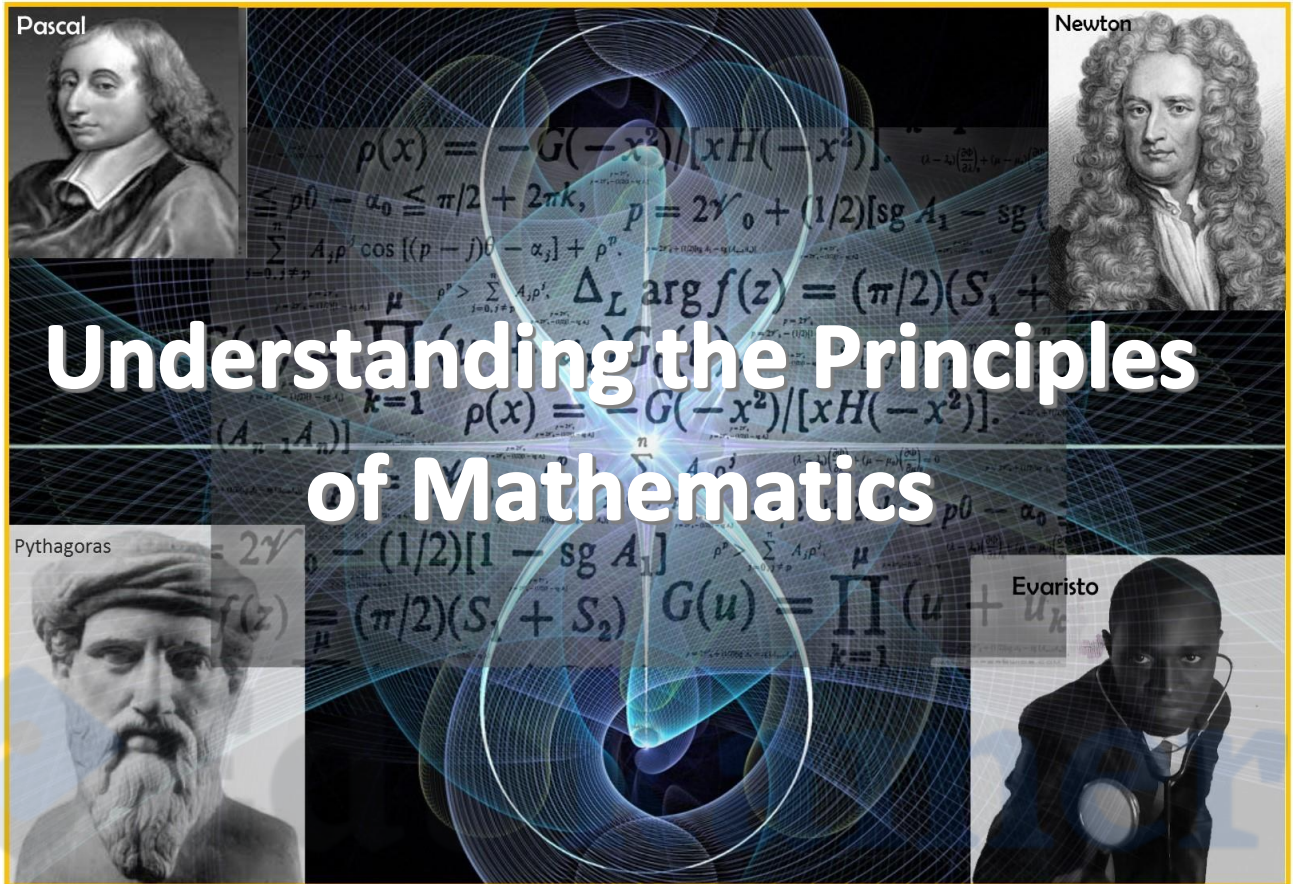
Evaristo Chanda

Understanding the Principles of Mathematics

'a small masterpiece'

Understanding the Principles of Mathematics

written by

Evaristo Chanda

published by

Glamorous Tuition Centre, School of intellectuals, Kitwe, Zambia

in conjuction with

GirlJam Business Services, Kitwe, Zambia

in

2016.

Preface

This book is intended for secondary school students studying mathematics both for the School Certificate and General Certificate of Education Ordinary Level (GCE). The material covered in this book are for Grade 10, Grade 11, Grade 12 and those sitting for the General Certificate of Education Ordinary Level (GCE). The book has been written in simple language to help students grasp and apply mathematical concepts easily. To help reinforce the material, the book is loaded with examples which have been explained in a classic and simple way. Most questions given as examples in this book are from past examination questions of the Examination Council of Zambia so are some of the tasks. This has been done to expose students early to Examination Question answering skills thereby boasting their confidence in the subject.

The book has covered at least 95% of the material tested in Mathematics syllabus D (Code 4024) Final Examination. Studying this material seriously help the student to yield great fruits in as far as Mathematics as a subject in concerned.

This book was written due to a request made by m students at Ordinary Level for a simplified mathematics book. Therefore the material contained in this book are also my teaching notes. Having taught mathematics both at ordinary and advanced level, I have come to understand most challenges students face in the subject. Based on this fact the material has been presented in a readable and understandable manner tackling these challenges in the process such that even the dullest student will appreciate mathematics.

Acknowledgement

My sincere gratitude goes to all who may have contributed to the production of this book whether directly or indirectly.

Most of the material contained in book has it origin in the past and has become an inheritance to everyone teaching mathematics.

Contents

1. Numbers

There different types of numbers: natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers and complex numbers.

Natural numbers are counting numbers that start from 1. It is the set N = {1, 2, 3, . . .}. When 0 is included in the set of natural numbers they are called whole numbers. Put simply, Whole numbers are natural numbers starting from 0. It is the set W = {0, 1, 2, 3, . . .}. Natural numbers that are divisible by 2 without leaving a remainder are called even numbers (e.g.2, 4, 6, . . .). Those that are not completely divisible by 2 are called odd numbers (e.g. 1, 3, 5, . . .).

Integers are positive and negative numbers and zero. It is the set Z = {. . . . . ., -1, 0, +1, . . .}.

Any number that can be written as a ratio is a rational number. In a rational number both the denominator (number below division sign in a fraction) and numerator (number above division sign in a fraction) are whole numbers. All whole numbers are rational number. Examples of rational numbers include $\frac{4}{5}$ and $\frac{7}{1}$.

An irrational number is any number that is not rational. An irrational can be written as a decimal, and not as a fraction. For example $\pi = 3.14159$............. and $\sqrt{2} = 1.41421$.........

All numbers (integers, rational numbers and irrational numbers) constitute the set of real numbers. Real numbers can be used to label every point on a continuous infinite line – the real line.

1.1 Common factors and Multiples

A number that cannot be divided exactly by any integer apart from one or itself is called a prime. A prime number is a positive integer. From the definition, 1 is not a prime number. Since 8 can be written as 2 x 4, it is not a prime. The numbers 2 and 4 are called its (prime) factors.

Factors are numbers that can be multiplied together to get another number. When factors of two or more numbers are compared, the factors that will be same to all to the numbers been compared are the common factors. As an example, let us compare the factors of 4 and 6.

4 : 1, 2, 4

6 : 1, 2, 3, 6

Since 1 and 2 are present in both sets of factors there are common factors of 4 and 6. However, it should be stated here that 1 is a common factor of every number.

A multiple is the product of a whole number and any natural number. Every whole number has an infinity (endless) number of multiples. That is it is not possible to list all the multiples of a whole number unless there are restrictions e.g. write down the first 6 multiples of 5. This is simple, multiply 5 by 1 through 6. The first 6 multiples of 5 are 5, 10, 15, 20, 25 and 30.

When multiples of two or more numbers are compared, the smallest multiple to all is called the lowest common multiple (LCM). The largest multiple to all is called highest common multiple (HCM).

For example. The comparing the multiples of 4 and 9

4: 4, 8, 12, 16, 20, 24, 28, 32, **36**, 40, 44

9: 9, 18, 27, **36**, 45

The LCM of 4 and 9 is **36**

1.2 Manipulation of numbers

Arithmetic is based on a few operations which are addition, subtraction, multiplication and division, satisfying a small number of rules. The rules governing manipulation of numbers can be summarised by the mnemonic **BODMAS**.

B – Brackets ( ) first

O – Of

D – Division

M – Multiplication

A – Addition

S – Subtraction (last)

Example

Evaluate the following:

a. $7 + 21 \div 3 \times 7 - 7$

$= 7 + 7 \times 7 - 7$

$= 7 + 49 - 7$

$= 56 - 7$

$= \textbf{49}$

b. $7 + (6 + 2) \div 4 \times 2$

$= 7 + 8 \div 4 \times 2$

$= 7 + 2 \times 2$

$= 7 + 4$

$= \textbf{11}$

Reasoning

a. **BODMAS. Since there are no brackets begin with division.**

b. **BODMAS**

Task

Evaluate the following

(a) $8 \div 4 + 2 \times 2$

(b) $-5 \times 3 + 6$

1.3 Handling fractions

A fraction or rational number is any quantity of the form

$$\frac{p}{q}$$

Where p, q are integers but q $\neq$ 0. It is of course essential that q $\neq$ 0, because division by zero is undefined.

p is called the numerator

q is the denominator

if $p \geq q$ the fraction is said to be improper, and if $p < q$ it is proper.

A number expressed in the form $3\frac{1}{2}$ (meaning $3 + \frac{1}{2}$) is called a mixed fraction. It is difficult to do calculations such as multiplication and division using mixed fractions. Such mathematical expressions must be avoided and should be written as a vulgar (improper) fraction $\frac{7}{2}$ instead, otherwise it might be mistaken for $3 \times \frac{1}{2}$.

Multiplication: Multiply numerator by numerator and denominator by denominator

Reciprocal: The reciprocal or inverse of a fraction is obtained by turning it upside down. E.g.

$$1 \Big/ \frac{4}{3} = \frac{3}{4}$$

Division: Dividing two fractions involves inverting the second fraction and then multiplying it with the first.

$$\frac{2}{3} \div \frac{3}{4}$$

$$= \frac{2}{3} \times \frac{4}{3}$$

$$= \frac{8}{9}$$

Addition and Subtraction

Find the common denominator which is the lowest common multiple. In case of two vulgar fractions the easiest way of finding the common denominator is by multiplying the two denominators.

(a) $\quad \frac{2}{3} + \frac{3}{4}$

$$= \frac{8 + 9}{12}$$

$$= \frac{17}{12}$$

(b) $\frac{1}{2} - \frac{3}{4}$

$$= \frac{2-3}{4}$$

$$= -\frac{1}{4}$$

BODMAS still applies when dealing with fractions

Examples

Evaluate the following

(a) $1\frac{2}{3} - 1\frac{2}{3} \times \frac{1}{4}$

$$= \frac{5}{3} - \frac{5}{3} \times \frac{1}{4}$$

$$= \frac{5}{3} - \frac{5}{12}$$

$$= \frac{20-5}{12}$$

$$= \frac{15}{12}$$

(b) $2\frac{5}{8} - 2\frac{1}{6} \div 1\frac{1}{12}$

$$= \frac{21}{8} - \frac{13}{6} \times \frac{13}{12}$$

$$= \frac{21}{8} - \frac{169}{72}$$

$$= \frac{189-169}{72}$$

$$= \frac{20}{72}$$

(c) $3 + \frac{1}{2} \div \frac{1}{4}$

$$= \frac{3}{1} + \frac{1}{2} \times \frac{4}{1}$$

$$= \frac{3}{1} - \frac{4}{2}$$

$$= 3 - 2$$

$$= 1$$

1.4 Significant figures

There are two types of numbers, exact and measured. Measured numbers are measured using a measuring instrument and these contain errors, e.g. 2.45 cm. Exact numbers are counting numbers and have no errors, e.g. 2 soccer balls.

Rules for significant figures

(i)   **Rule 1**: Zeros at the beginning of a number are not significant. They only help to locate the decimal point. E.g. 0.0024kg has 2 significant figures.

(ii)   **Rule 2**: Zero(s) in the middle of numbers are always significant. E.g. 7.002kg has 4 significant figures.

(iii)   **Rule 3**: Zeros at the end of a number and before an implied decimal point may or may not be significant.

(iv)   **Rule 4**: Zeros at the end of a number and after the decimal point are significant. E.g. 457.1000 has 7 significant figures.

(v)   **Rule 5**: all non-zero digits are significant.

1.5 Scientific notation

Scientific notations are used make manipulation of cumbersome numbers easy. That is scientific notation is used to simplify a number which is very large or too small. A scientific notation sometimes called standard form is an expression of the form

$$a \times 10^n$$

where **a** is a number between 1 and 10, **n** is an integer and **10** is the base.

To express a number in standard form, first identify where the decimal point is and move it until the number is between 1 and 10. Count the number of times you moved the decimal point to get the value of the exponent (**n**). When the original number is more than 1 the exponent is positive. The exponent is negative when the original number is less than 1. Round off the figure to the required number of significant figures.

Writing a number that is less than one in standard form. For example, 0.0000000000601

0.0 0 0 0 0 0 0 0 0 6 0 1

The decimal point was moved 11 times to make the number be between 1 and 10. Since the original number was less than 1 the exponent is negative. The number becomes

$$6.01 \times 10^{-11}$$

Writing a number which is more than one in standard form. E.g. 60200000000000000

6 0 2 0 0 0 0 0 0 0 0 0 0 0 0 0

The decimal point in this case was after the last digit. It was moved 16 times to make the number be between 1 and 10. The exponent is positive because the original number was greater than 1. The number becomes

$$6.02 \times 10^{16}$$

Example 1. The population of Zambia in 2010 was 13 046 508. Express this number in standard form correct to 3 significant figures.

> **Solution**
>
> 1.30×10^7

Example 2. Express the number 2595455000 in standard form correct to 3 significant figures.

> **Solution**
>
> 2.60×10^9

1.6 Calculations involving scientific notations

Multiplication

- Multiply the coefficients together
- Add the exponents
- Do not change the base

For example, $(4 \times 10^{11}) \times (2 \times 10^{-4}) = \mathbf{8 \times 10^7}$

$$(3 \times 10^7) \times (2 \times 10^9) = \mathbf{6 \times 10^{16}}$$

$$(6 \times 10^{11}) \times (4 \times 10^6) = 24 \times 10^{17} = \mathbf{2.4 \times 10^{18}}$$

Note that the answer 24 x 10¹⁷ is wrong even though right because it is not in standard form.

Division

- ✓ Divide the coefficients
- ✓ Subtract the exponents
- ✓ Do not change the base

For example

$$\frac{6 \times 10^9}{3 \times 10^6} = 2 \times 10^3$$

Addition and subtraction

If the exponents are the same, simply add or subtract the coefficients and do not change the base and the exponent.

For example

$$(2 \times 10^4) + (6 \times 10^4) = \mathbf{8 \times 10^4}$$

$$(5 \times 10^6) - (2 \times 10^6) = \mathbf{3 \times 10^6}$$

If the exponents are not the same, move the decimal point of the number with the smallest exponent to make the exponents equal. Then, add or subtract as above and write the final answer in standard form.

e.g. $(2 \times 10^5) + (3 \times 10^7) = (0.02 \times 10^7) + (3 \times 10^7)$

$$= \mathbf{3.02 \times 10^7}$$

1.7 Fractions, decimal numbers and percentages

Fractions and decimal numbers can be converted into percentage. The symbol (%) is used to represent a percentage.

To convert a decimal or fraction into a percentage just multiply with 100%. For example,

(a) $0.25 = 0.25 \times 100\% = 25\%$
(b) $0.00056 = 0.00056 \times 100\% = 0.056\%$
(c) $1 = 1 \times 100 = 100\%$

(d) $\frac{1}{4} = \frac{1}{4}$ x 100% = 25%

To convert a percentage into a fraction or decimal just divide by 100%. For example,

(a) $50\% = \frac{50\%}{100\%} = \frac{1}{2} = \mathbf{0.5}$

(b) $75\% = \frac{75\%}{100\%} = \frac{3}{4} = \mathbf{0.75}$

Note that the maximum normal percentage is 100. Any percentage above 100 is abnormal. However, it is possible to have an abnormal percentage. To give an example, a sales man invests k50 in a certain item. After selling the item he earns k150. One can say the sales man has made an abnormal profit. A profit of K100 in this case represents 200% which is abnormal.

1.8 Rounding off and Manipulation of Decimal Numbers

When rounding off, look at the figure to the right of the figure that is to be at end and follow the following rules

(a) If it is more than 5, increase by 1 the number before it (the number to be rounded), and drop it and all the digits after it.
(b) If it is less than 5, drop it and all the digits after it.
(c) If it is 5, round the number so that it will be even. Zero is considered even when rounding off.

Examples

1. Round 826.835 to five significant figures. Look at the sixth figure. It is a 5, so now you must look at the fifth also. That is a 3, which is an odd number, so you round the original number up to 826.84
2. Round 54.8514 to three significant figures. Look at the fourth figure. It is a 5, so now you must also look at the third figure. It is 8, an even number, so you simply drop the 5 and the figures that follow it. The original number becomes 54.8.
3. Round 4.0495 to four significant figures. Look at the fifth figure. It is a 5, so now you must look at the fourth figure, it is 9. The nine rounds off to a ten not a zero. The original number becomes 4.050.
4. Round 62.5347 to four significant figures. Look at the fifth figure. It is a 4, a number less than 5. Therefore, you must simply drop every figure after the fourth figure digit, and the original number rounds off to 62.53.
5. Round 3.78721 to three significant figures. Look at the fourth figure. It is a 7, a number greater than 5, so you round the original number up to 3.79.

Addition and subtraction of decimal numbers

The calculated value should have the same number of digits to the right of the decimal point as that of the number with the fewest digits used in the calculation.

Example. $3.441 + 2.24 + 12.2$

Here the number 12.2 has the fewest number of digits to the right of the decimal point, and it will determine the number of digits to the right of the decimal point in the final answer.

$3.441 + 2.24 + 12.2 = 17.881 = \mathbf{17.9}$

Multiplication and division of decimal numbers

The calculated value should have the same number of significant figures as that of the quantity with the fewest number of significant figures used in the calculation.

Example. 2.11 x 3.233

Here the number 2.11 has the fewest number of significant figures, and it will determine the number of significant figures in the final answer.

$2.11 \times 3.233 = 6.82163 = \mathbf{6.82}$

To divide decimal numbers without a calculator, convert the decimal numbers to whole numbers and simplify.

Example. $2.16 \div 0.03$

(b) $2.16 \div 0.03$

$= \dfrac{2.16}{0.03} \times \dfrac{100}{100}$

$= \dfrac{216}{3}$

$= \mathbf{72}$

(a) $4.555 \div 0.05$

$= \dfrac{4.555}{0.005} \times \dfrac{1000}{1000}$

$= \dfrac{4555}{5}$

$= \mathbf{911}$

If the decimal numbers have 1 decimal place (dp) multiply by 10, 2dp by 100, 3dp by 1000 etc.

Dealing with combined addition or subtraction and multiplication or division problems

The first step is to apply the rules for addition or subtraction, determine the number of significant figures for that step, then apply the rules for multiplication or division.

Task

1. write $\dfrac{3}{8}$ as a decimal fraction
2. find the exact value of
   (a) 2.401 x 0.03
   (b) 1.692 + 4.52
3. Express 0.52 as a fraction, giving your answer in its lowest terms.
4. express $\dfrac{21}{40}$ as a percentage
5. express 4 995 257 in scientific notation correct to 3 significant figures
6. the number of orphans and vulnerable children in one province of Zambia is 599 900
   (a) Write down this number in standard form.
   (b) Express this number correct to 1 significant figure.

1.9 Ratio and proportion

Ratios are used to compare two quantities. For example, a loaf of bread cost K7. This is comparing the number of loafs of bread to cost. Ratios can be expressed in three ways. Using the example above, the ratio of loaf of bread to cost can be written as:

(a) $1 : 7$

(b) $\frac{1}{7}$

(c) 1 to 7

A proportion is an expression that equates one ratio to another. A proportion joins two ratios by an equal sign.

$$\frac{p}{q} = \frac{r}{s}$$

This does not mean p = r or q = s.

Proportions can be manipulated easily by cross multiplying.

Direct and inverse proportion

When an increase in one quantity causes and increase in another quantity and vice versa such that the ratio between the two quantities is constant, then the two quantities are in direct proportion. Thus if a and b are direct proportional (a α b), then

$$\frac{a}{b} = k$$ where k is a constant.

Two quantities are inversely proportional when one quantity increases causes the other quantity to decrease such that the product of their corresponding values is constant.

Thus if a and b are inversely proportional (a $\alpha \frac{1}{b}$), then

$$ab = k$$

Example 1.

Chomba planned to employ 20 men to build his house in 7 days. On the day work was to start, he decided to reduce the number of men so that work could now be completed in 28 days, working at the same rate.

(a) how many men were needed for the work?

(b) express the number of men that completed the work in 28 days to the number of men he planned to employ as a ratio in its simplest form.

Solution and Reasoning

This is an inverse proportion problem, because when the number of men reduces the working days increases.

(a) Let x be the number of men to complete the work in 28 days, and express these relations in direct proportion.

Number of Men		days
20	→	7
x	→	28

now interchange either the number of men or days, here we interchange days.

| 20 | → | 28 | |
| x | → | 7 | cross multiply and solve for x |

$$x = \frac{20 \text{ men } \times 7 \text{ days}}{28 \text{ day}} = \textbf{5 men}$$

(b) $5 : 20$

$= \textbf{1 : 4}$

Example 2

In a certain school, the ratio of boys to girls is 3 : 5. Given that there are 600 girls, calculate the total number of pupils in the school.

Solution and Reasoning

Let y represent the total number of pupils in the school. In this ratio 5 represent girls and the total   ratio is 8 (i.e. $3 + 5$). Therefore, an expression involving the number girls is written as

$$\frac{5}{8}y = 600 \qquad\qquad \text{now solve for y by cross multiplying}$$

$$5y = 8 \times 600$$

$$5y = 4800 \qquad\qquad \text{divide by 5 on both sides}$$

$$y = 960$$

Task

Three candidates A, B and C took part in a student's union presidential elections. Candidate A received 12 600 votes and candidates B and C followed each other respectively.

(i)   given that the ratio of the votes of candidate A to that of Candidate B is 3 : 2, calculate the number of votes received by Candidate B,

(ii)   given also that Candidate B received 600 more votes than Candidate C, calculate the number of votes for Candidate C.

2. Powers and indices

In certain situations it is required that a number is multiplied by itself a couple number of time. For example multiplying 3 by itself 10 times will be 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3. This looks cumbersome, however, powers and indices provide a simpler way of multiplying a quantity by itself given a number of times. Using powers the expression above can be written as 3^{10}, which is less cumbersome. Let us observe the pattern for powers using 3 as an example.

$$3 \text{ x } 3 = 3^2$$

$$3 \text{ x } 3 \text{ x } 3 = 2^3 \text{ , etc}$$

For a given number p we have

$$p^n = p \text{ x } p \text{ x } \ldots \text{ x } p$$

Here p is called the base, n is the power or index.

Indices makes it easy to manipulate quantities with the same base.

2.1 Rules of indices

1. $p^0 = 1$ (any number raised to the power 0 is 1)

2. $p^1 = p$ (any number raised to the power 1 is that number)

3. $p^m \text{ x } p^n = p^{m+n}$

4. $\dfrac{p^m}{p^n} = p^{m-n}$

5. $(p^m)^n = p^{mn}$

6. $(pq)^n = p^n q^n$

7. $1^n = 1$ (1 raised to any power is 1)

2.2 Square and Cube roots

The nth root of a number p is denoted by

$$\sqrt[n]{p} \quad , \quad \text{it is true that} \quad \sqrt[n]{p} = p^{1/n}$$

When n = 2, it is called square root (also represented by the symbol $\sqrt{p}$ or $p^{1/2}$). If **a** is a square root of **b**, then $a^2 = b$.

For example, 2 is the square root of 4 because $2^2 = 4$, denoted as $\sqrt{4} = 2$.

It is not possible to have a negative square root. That is an even number has to be positive to yield a real root.

When n = 3, it is called cube root. The cube root of a number **p**, is any number when raised to the power 3 yields the number **p**.

For example, 2 is the cube root of 8 because $2^3 = 8$. Also denoted as $\sqrt[3]{8} = 2$. Also, - 2 is the cube root of -8, because $-2^3 = -8$. This is denoted as $\sqrt[3]{-8} = -2$. Thus, it is possible to have a negative cube root of a negative number.

2.3 Surds

Surds are quantities that contain square roots of prime numbers e.g. $\sqrt{2}$ and $\sqrt{3}$.

Example1

Given that 4^{-2} is the reciprocal of 4^n, find the value of n.

> **Solution**
>
> If 4^{-2} is the reciprocal of 4^n, then
>
> $$4^n = \frac{1}{4^{-2}}$$
>
> $$4^n = 1 \div \frac{1}{4^2}$$
>
> $$4^n = 1 \times 4^2$$
>
> $$4^n = 4^2$$
>
> Since the bases are the same, **n = 2**

Example 2

Evaluate $5^3 \times 5^{-1} \times 8^0$

> **Solution**
>
> Express everything to the same power, then apply multiplication rule of indices (add the powers). Note that $8^0 = 1$ which is the same as $5^0 = 1$.
>
> $5^3 \times 5^{-1} \times 5^0$
>
> now add the powers while maintaining the base
>
> $5^{3 + (-1) + 0} = 5^2$
>
> $= \mathbf{25}$

3. Set theory

A set is a collection of well-defined objects. These objects are called elements or members.

The elements of a set are usually denoted by small letters or numbers while Capital letters are used to name the sets. E.g. set A = {a, b, c, d} or set B = {1, 2, 4}. This method of listing elements is called roster.

If an element is a member of set, e.g. **a** is a member of set **A**, this is written as **a** $\in$ **A.**

If an element is not a member of a set, e.g. 2 is not a member of set A, this is written as **2** $\notin$ **A.**

3.1 Set builder notation

Sometimes it is not possible to list all the members of a set. Set builder notation is a simple way of representing the members of a set when they are too many to be listed. Set builder notation describes the members of a set rather than listing them. For example

Set B = {a, b, c, d, e, f, g, h, I, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x. y, z} in roster form.

B= {x | x is a letter of the English alphabet} in set builder notation.
This is read as B is a set of all x such that x is a letter of the English alphabet.

3.2 Some important sets

N = {0, 1, 2, 3, . . .}, the set of natural numbers.

Z = {. . ., -1, 0, 1, . . .}, the set of integers.

Z+ = {1, 2, 3, 4, . . .}, the set of positive integers.

Q = {a/b | a $\in$ Z, b $\in$ Z, b $\neq$ 0}. The set of rational numbers.

R = {x | x is a real number}, the set of all real numbers.

3.3 set concepts

The empty set
This is a set that has no elements. Its symbol is { } or Ø

The cardinality of a set
The cardinality of a set in the number of elements in a set. Let set A = {1, 3, 4, 6}. The cardinality of set A denoted as n(A) = 4.

Subsets
A subset is a set contained in another set. For example, set A = {1, 3, 4, 6}. If we form another set B = {3, 6} using the members of set A, then set B is a subset of A. Symbolically this is denoted as B $\subset$ A.
Moreover, it should be noted that every set is a subset of itself, and that an empty set is a subset of every set.

The power set

The power set of a set refers to the number of subsets contained in that set. Let set B = {3, 6}, the power set of B denoted as P(B) is the set of all subsets of set B. The subsets of set B are 4. These are { }, {3, 6}, {3} and {6}.
To find the number of subset of a set use the formula

$$\boxed{\text{Number of subsets} = 2^n}$$

Where n is the number of elements.

Number of elements	Number of subsets
0	1
1	2
2	4
3	8

Equality of sets

Two or more sets are equal if and only if they contain the same elements even if they are not arranged in the same order. If set A = {a, b, c}, set B = {c, a, b} and set C = {a, b}, then set A = set B but set A ≠ set C. Also set B ≠ set C.

The union set

Let set X = {1, 2, 4, 5, 7} and set Y = { 2, 3, 6, 7} then the union set of X and Y denoted as X U Y is the set containing all the elements of set X and Y. Thus X U Y = {1, 2, 3, 4, 5, 6, 7}.
Note that same elements from both sets are only written once.

The intersection set

Let set X = {1, 2, 4, 5, 7} and set Y = { 2, 3, 6, 7} then the intersection set of X and Y denoted as X ∩ Y is the set containing the elements that are same to both set X and Y. thus X ∩ Y = {2}.

Disjoint sets

Two or more sets are disjoint if they do not contain at least 1 common element. That is they have no intersection set. e.g. A = {1, 2, 3} and B= {a, b, c} are disjoint sets.

The universal set

This is the main set from which all other sets are derived. It is represented by the symbols E or U.
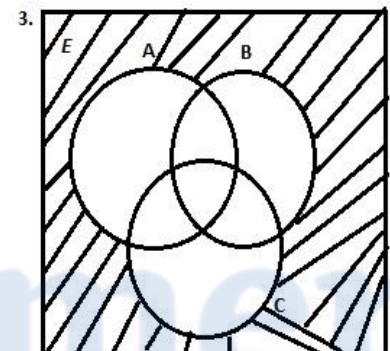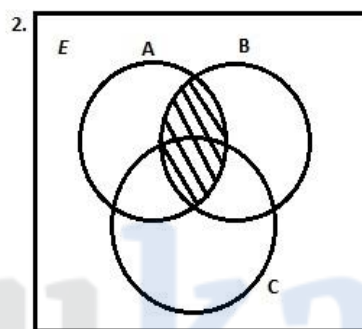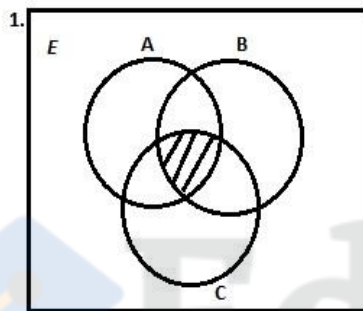
The complement of a set

Let the universal set E = {a, b, c, d, e, f, h} and set A = {a, e, h}. Then the complement of set A (denoted as A'} is the set of all elements that are not in set A but are in the universal set. Thus A' = {b, c, d, f}.

Venn diagram

Venn diagram display the relationships between two or more sets.
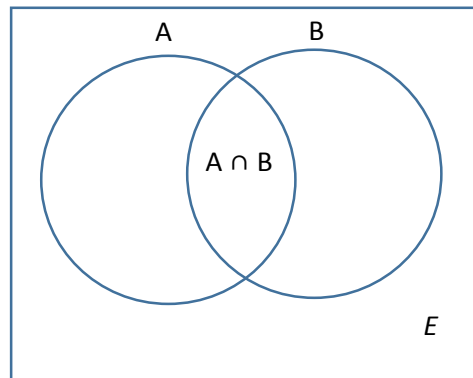
Describing shaded region on a Venn diagram
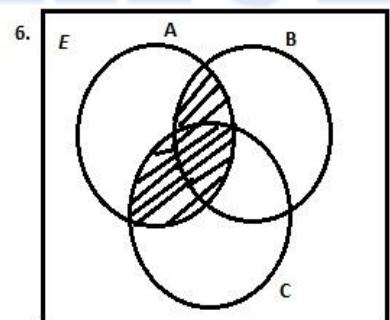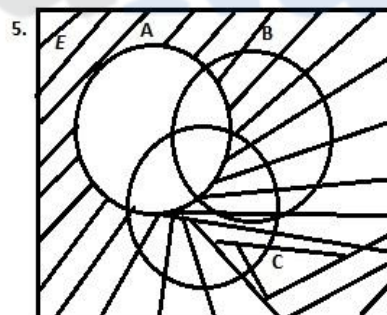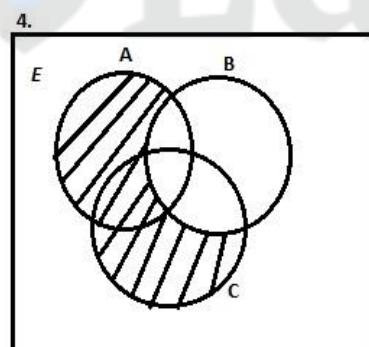
1. $A \cap B \cap C$ 2. $(A \cap B \cap C) \cup (A \cap B)$ 3. $(A \cup B)' \cap C'$ 4. $B' \cap (A \cup C)$

5. A' 6. $(A \cap B \cap C) \cup (A \cap B) \cup (A \cap C)$

Example 1
Given that $E = \{x : -8 \leq x \leq 8, x \in \text{integer}\}$ and A, B and C are subsets of E, such that
A = $\{x : -7 < x < 7\}$,
B = $\{x : -5 < x \leq 1\}$
C = $\{x : -1 \leq x < 4\}$
List the elements of
    (i)   B ∩ C   [1]
    (ii)   A ∩ (B U C)'   [2]

Solution

E = {-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8}   B UC = {-4, -3, -2, -1, 0, 1, 2, 3}
A = {-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6}   (B U C)' = {-8, -7, -6, -5, 4, 5, 6, 7, 8}
B = {-4, -3, -2, -1, 0, 1}
C = {-1, 0, 1, 2, 3}

    (i)   B ∩ C = {-1, 0, 1}
    (ii)   A ∩ (B U C)' = {-6, -5, 4, 5, 6}

Example 2
In a class of 40 pupils, 16 like physics, 17 like mathematics, 24 like chemistry and the rest like other subjects not stated.
5 like all the three subjects
4 like mathematics and physics only
6 like physics and chemistry only
3 like mathematics and chemistry only

    (a) Illustrate this information on a clearly labelled Venn diagram.   [3]
    (b) Hence or otherwise find
        (i)   The number of pupils like one subject only,   [1]
        (ii)   The number of pupils who do not like any of the three subjects.   [2]

Solution

    (a)
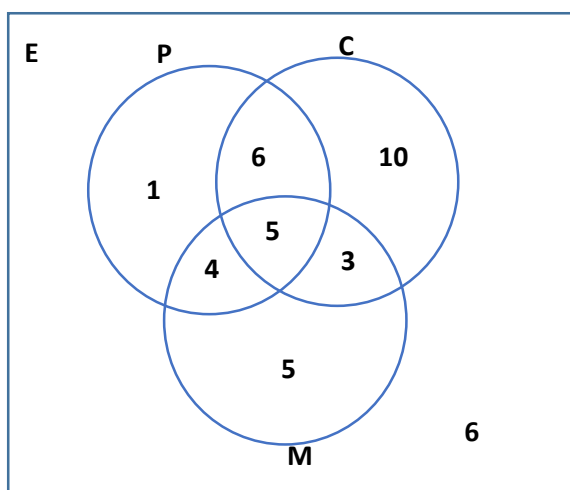
    (b) (i) $10 + 5 + 1 = 16$ (ii) 6

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Reasoning

(a) Since some students liked all three subjects, they has to be an intersection for all three sets. Fill this first, followed by the intersections of two sets only. Note that numbers e.g. 16 like physics include even those who liked chemistry and mathematics. In shorts, the whole circle for physics should add up to 16. This applies to other circles. Add all the numbers in the three circles once then subtract from 40 to find those who liked other subjects not stated.

(b) Use the vein diagram

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Example 3

At one college, a group of 25 students were asked which Cell Phone service providers they subscribed to. The results are shown in the Venn diagram below.
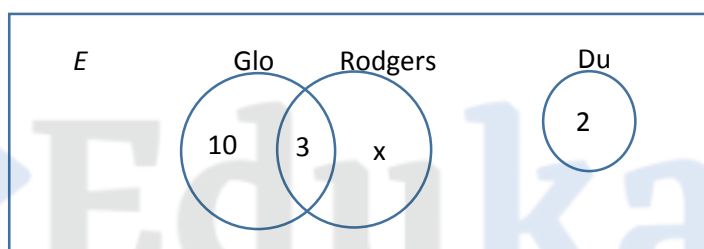
(i)   Calculate the value of x.   [1]
(ii)   Given that G = {Glo}, R = {Rodgers} and D = {Du}, find,
        (a) n(G ∩ R)   [1]
        (b) n(D U G').   [2]

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Solution

(i) $10 + 3 + 2 + x = 25$
        $15 + x = 25$
        $x = 25 - 15$
        $x = 10$
(ii)   (a) n(G ∩ R) = 3
        (b) n(D U G') = 10 + 2 = 12

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Reasoning

(i)   Add all the numbers in the three sets, equate to 25 and solve for x.
(ii)   (a) n(G ∩ R) means the number of elements in G ∩ R
        (b) n(D U G') means the number of elements in D U G'. cancel set G, then add the numbers remaining.i.e. x + 2, since x= 10 it becomes 10 + 2 = 12

</div>

Example 4

$E = \{x : x < 10, \in \text{Natural numbers}\}$
$P = \{x : x \text{ is a prime number}\}$
$Q = \{x : x \text{ is an even number}\}$

(i)   list down the elements of:
    (a) P.   [1]
    (b) Q.   [1]
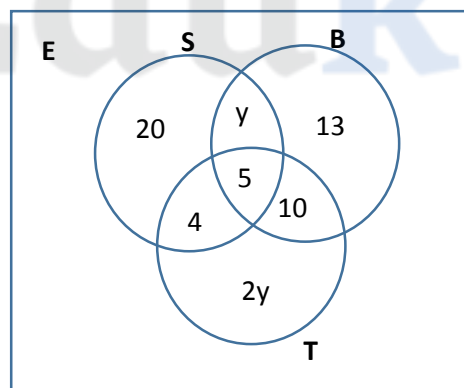(ii) Find n(P ∩ Q).   [1]

Solution
(i)   (a) P = {2, 3, 5, 7}
(b) Q = {2, 4, 6, 8}
(ii)   n(P ∩ Q) = 1

Reasoning
$E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
1 is not prime number

Example 5
Tokhozani Sports club offers Squash (S), Badminton (B0 and Tennis (T). the Venn diagram below shows choices of the 73 members of the club.

(i)   Calculate the value of y.   [2]
(ii)   Find the number of members who played squash or Tennis but not Badminton.   [1]
(iii) How many members played two different sports only?   [1]
(iv) Find the number of members who played one sport only.   [1]

Solutions

(i) $20 + 13 + 10 + 4 + 5 + y + 2y = 73$
        $52 + 3y = 73$
        $3y = 73 - 52$
        $3y = 21$
        $y = 7$
(ii) $20 + 4 + 14 = $ **38 members**
(iii) $4 + 10 + 7 = $ **21 members**
(iv) $20 + 13 + 14 = $ **47 members**

Reasoning

After solving for y, the Venn diagram
Becomes as shown above

Task

A class of 41 girls takes History (H), Commerce (C) and Geography (G) as optional subjects. The Venn diagram below shows their choice distribution.
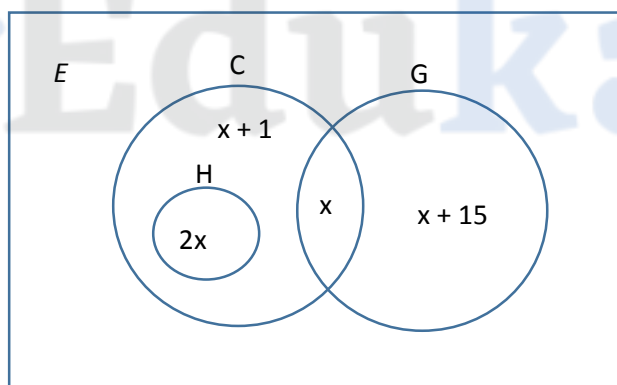
(i)   Calculate the value of x.   [2]
(ii)   Find
        (a) n(H U G),   [1]
        (b) n(G' ∩ H').   [1]

Solution

(i) $x + 1 + 2x + x + x + 15 = 41$
$$16 + 5x = 41$$
$$5x = 41 - 16$$
$$5x = 25$$
$$\mathbf{x = 5}$$

(ii)   (a) $n(H \cup G) = 2x + x + x + 15$
$$= 2(5) + 5 + 5 + 15$$
$$= \mathbf{35}$$
         (b) $n(G' \cap H') = x + 1$
$$= 5 + 1$$
$$= \mathbf{6}$$

Reasoning

i.   Add everything in all the 3 sets, equate to 25, then solve for x.

ii.   (a) add the numbers in set H and G after replacing x with the calculated value
(b) cancel out set H and G, what remains is G' ∩ H'.

Task

The diagram below shows three sets A, B and C.
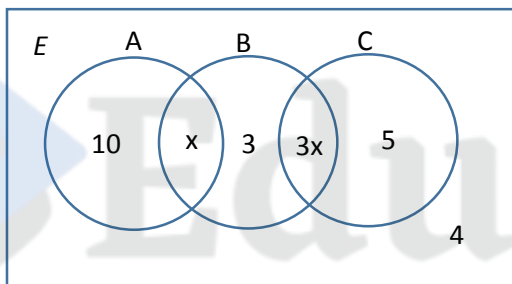
Given that $n(A \cup B \cup C) = 50$, find

     (i)   the value of x,   [2]
     (ii) $n(A \cup B)$,   [1]
     (iii) $n(B \cup C)'$,   [1]
     (iv) $n(A' \cap C')$.   [2]

4.Algebra

Algebra is the branch of mathematics that uses letters (symbols) and numbers to perform arithmetic. Normally, letters x, y and z represent variables, which may take a range of values: a, b and c represent constants (quantities with fixed values). Other small letters such l, m and n may be used to represent integers or whole numbers.

4.1 Important algebraic rules for linear functions

The distributive rule

(i) $a(c + d) = ac + ad$

e.g. $2x(2 - x) = \mathbf{4x - 2x^2}$

(ii) $(a + b)(c + d) = a(c + d) + b(c + d)$ this is called expanding
$$= ac + ad + bc + bd$$

e.g. $(3x - 2)(x + 1) = 3x(x + 1) - 2(x + 1)$
$$= 3x^2 + 3x - 2x - 2$$
$$= \mathbf{3x2 + x - 2}$$

(iii) $(a + b)(a - b) = a(a - b) + b(a - b)$
$$= a^2 - ab + ab - b^2$$
$$= \mathbf{a^2 - b^2} \quad \text{(this is called the difference of two squares)}$$

(iv) $(a + b)^2 = (a + b)(a + b)$
$$= a(a + b) + b(a + b)$$
$$= a^2 + ab + ab + b^2$$
$$= \mathbf{a^2 + 2ab + b^2}$$

The associative rule

$a(bc) = (ab)c$

e.g. $x(2y) = (2x)y$
$$= \mathbf{2xy}$$

Commutative rule

$ab = ba$ if ab = 0, then both a and b must be zero

4.2 Factorization

The reverse of expansion is called factorization. This involves the splitting of an expression into factors. Examples are shown below

(a) $mn - km - hn + hk$
$$= (mn - km) - (hn - hk)$$
$$= m(n - k) - h(n - k)$$
$$= \mathbf{(m - h)(n - k)}$$

(b) $5 - 20x^2$

$= 5(1 - 4x^2)$

$= 5(1^2 - 2^2x^2)$

$= \mathbf{5(1 - 2x)(1 + 2x)}$

(c) $4 - 16x^2$

$= 4(1 - 4x^2)$

$= 4(1^2 - 2^2x^2)$

$= \mathbf{4(1 - 2x)(1 + 2x)}$

(d) $m^2 - 4n^2$

$= m^2 - 2^2n^2$

$= \mathbf{(m - 2n)(m + 2n)}$

(e) $3y^2 - 12$

$= 3(y^2 - 4)$

$= 3(y^2 - 2^2)$

$= \mathbf{3(y - 2)(y + 2)}$

(f) $x^2 - 4$

$= x^2 - 2^2$

$= \mathbf{(x - 2)(x + 2)}$

Remember

The

Difference

Of Two

Squares

4.3 Factorizing quadratic functions

These are functions in which the highest power of a variable is 2. It is an expression of the form $ax^2 + bx + c$.

(a) $2x^2 - x - 3$

- ✓ First find the factors, using the product and sum.
- ✓ To find the product multiply the constant and the coefficient of x^2, here the product is -6 (i.e. 2 x -3).
- ✓ The sum is the coefficient of the term in x, here the sum is -1.
- ✓ Find two factors (numbers) such that when you multiply them you will get the product and when you add them you will get the sum. These numbers are usually factors of the product.
  The factors here are – 3 and 2.
- ✓ Replace the mid-term (term in x) with these factors as coefficients of x in an order that favours factorization. That is
  $2x^2 + 2x - 3x - 3$
- ✓ Finally factorize the expression
  $= (2x^2 + 2x) - (3x + 3)$
  $= 2x(x + 1) - 3(x + 1)$
  $= (2x - 3)(x + 1)$
- ✓ Thus $\mathbf{2x^2 - x - 3 = (2x - 3)(x + 1)}$

(b) $p^2 + 2p - 3$

Product = - 3, sum = 2 and factors are 3 and -1

$= p^2 - p + 3p - 3$

$= (p^2 - p) + (3p - 3)$

$$= p\,(p - 1) + 3\,(p - 1)$$
$$= (p + 3)\,(p - 1)$$

It should be noted that some expressions cannot be factorized. e.g. $x^2 + x - 1$

4.4 Algebraic fractions

Algebraic fractions are manipulated in the same way as normal fractions.

(a) Express $\dfrac{x+2}{3} - \dfrac{2x-3}{4}$ as a single fraction in its simplest form.

Find the common denominator (just multiply the two denominators) and manipulate as normal fractions

$$\frac{x+2}{3} - \frac{2x-3}{4} = \frac{4\,(x+2) - 3\,(2x-3)}{12}$$

$$= \frac{4x + 8 - 6x + 9}{12} \qquad \text{collect like terms and simplify}$$

$$= \frac{17 - 2x}{12}$$

(b) Express $\dfrac{5}{2x-1} - \dfrac{7}{3x-2}$ as a single fraction in its simplest form.

$$\frac{5}{2x-1} - \frac{7}{3x-2} = \frac{5\,(3x-2) - 7\,(2x-1)}{(2x-1)\,(3x-2)} \qquad \text{multiply the two denominators to find the common}$$

Denominator. Do not expand.

$$= \frac{15x - 10 - 14x + 7}{(2x-1)\,(3x-2)}$$

$$= \frac{x - 3}{(2x-1)\,(3x-2)}$$

(c) Express $\dfrac{2a}{x-1} - \dfrac{a}{x-2}$ as a single fraction in its simplest form.

$$\frac{2a}{x-1} - \frac{a}{x-2} = \frac{2a\,(x-2) - a\,(x-1)}{(x-1)\,(x-2)} \qquad \text{multiply the two denominators to find the common}$$

Denominator. Do not expand.

$$= \frac{2ax - 4a - ax + a}{(x-1)\,(x-2)}$$

$$= \frac{ax - 3a}{(x-1)\,(x-2)}$$

$$= \frac{a\,(x-3)}{(x-1)\,(x-2)}$$

Task

1. Express $\dfrac{x-1}{x^2-3x+2}$ as a simplified single fraction.

2. simplify $\dfrac{3y^2 - 5y - 12}{y^2 - 9}$

4.5 System of Simultaneous linear equations in two variables

The variables commonly used are x and y. These systems of equations can be solved using the addition or subtraction method, substitution method or using matrices. However, in this section we will focus on the first two. The matrix method will be dealt with later in the chapter on matrices. Examples are given below.

(a) **Solve the simultaneous equations**

$y - 7 = 1$
$2x - y = 2$

Here we use the substitution method (it involves making one of the variables subject and substituting it in the other equation so that only one variable remains for easy manipulation).

First name the equations (i) and (ii)
$y - 7 = 1$ ....(i)
$2x - y = 2$ ....(ii)

Since equation (i) has only one variable, solve for this valuable
$y - 7 = 1$
$y = 8$
Substitute this value of y in equation (ii) then solve for x
$2x - 8 = 2$
$2x = 10$
$x = 5$

(b) Solve the system of equations below

$2x + 5y = 16$
$3x - 2y = 5$

Here we use the subtraction method. Equations should be expressed in such a way that when they are added or subtracted one variable will be eliminated. This may involve multiplying both equations by constants.

Name the equations (i) and (ii)
$2x + 5y = 16$ ....(i)
$3x - 2y = 5$ ....(ii)

Here it is not easy to make one of the variables subject, therefore we will eliminate one variable (x) by first multiplying equation (i) by 3 and equation (ii) by 2 and then subtracting equation (ii) from equation (i).

$$
\begin{array}{cc}
3 & 2x + 5y = 16 \quad ....(i) \\
2 & 3x - 2y = 5 \quad ....(ii)
\end{array}
$$

We get

$\quad\quad 6x + 15y = 48$ ....(iii)
$- \quad 6x - 4y = 10$ ....(iv)
$\quad\quad\;\; 0 + 19y = 38$ ....(v)

Solve equation (v) for y
$19y = 38$
$y = 2$
Substitute this value of y in equation (i) then solve for x
$2x + 5(2) = 16$

$2x + 10 = 16$

$$2x = 6$$
$$\mathbf{x = 3}$$

The method of choice when solving simultaneous equations depend on the question. Even though all the methods can be used to solve equations, there is one which is the simplest. Before answering look closely at the question and choose a simpler method. This means students should know at least the two methods explained in this chapter.

Task

Solve the systems of equations below

1. $x + y = 0$
   $x - y = 2$

2. $3x - y = 13$
   $2y - y = -1$

4.6 Inequations

These are expressions that uses inequality signs ($>$, $<$, $\leq$ or $\geq$) in place of the equal sign. $>$ - greater than, $<$ - less than, $\leq$ - less than or equal to and $\geq$ - greater than or equal to.

Inequations are solved in the same way as equations. However, for inequalities the sign changes when you divide by a negative number.

Examples

(a) Solve the inequation $4b - 3 < 6b + 4$
   Collect like terms and simplify
   $4b - 6b < 3 + 4$
   $-2b < 7$
   $\mathbf{b > -\dfrac{7}{2}}$ note how the sign changes after dividing by a negative number

(b) Solve $-5x - 9 \leq 16$
   $-5x - 9 \leq 16$
   $-5x \leq 16 + 9$
   $-5x \leq 25$
   $\mathbf{x \geq 5}$

(c) Solve the inequation $7 + 2t < 9$
   $7 + 2t < 9$
   $2t < 9 - 7$
   $2t < 2$
   $\mathbf{t < 1}$

(d) find the integer n such that $n + 3 < 11 < n + 5$
   This expression is made up 2 inequalites, which are n + 3 < 11 and 11 < n + 5, solve them separately.
   $n + 3 < 11$ $n + 5 > 11$
   $\mathbf{n < 8}$ $\mathbf{n > 6}$
   Therefore, **n = 7. That is the integer 7 satisfies the two conditions above.**

Task

1. solve the inequality $9t - 4 < 12t - 10$
2. given that $-2 \leq x \leq 3$ and $-7 < y < 4$, find
    (a) the largest possible integer value of $x - y$
    (b) the smallest possible value of $x + y$
    (c) the value of y if $y^2 = 36$

4.7 Changing subject of the formula

Changing subject of the formula means forming an equation of the type

$$a = \frac{F}{m}$$

Here **a** is the subject.

To change the subject (e.g. to F or m) just manipulate as normal algebraic arithmetic

Thus, $F = ma$ and $m = \frac{F}{a}$

Example

Given that $d = \frac{p+y}{5y}$, express y in terms of d and p.

This means make y subject of the formula. to do this express both sides as fractions and cross multiply

$\frac{d}{1} = \frac{p+y}{5y}$ after cross multiply we get

$5dy = p + y$ collect all the terms in y to one side

$5dy - y = p$ factor out y

$y(5d - 1) = p$ now divide by 5d – 1 on both sides of the equation

$y = \frac{p}{5d-1}$

Task

Given that $y = \sqrt{x - 9}$, expess x in terms of y

5.Relations and Functions

A relation is an association between members of two sets. Elements in two sets are linked by certain relation. The first set (input) is called the domain and the second set (output) is called the range.

A relation can be many to one, one to many, many to many or one to one. In a many to one, at least two elements of the set of inputs are mapped onto the same element in set of outputs. In a one to one, an element of the inputs maps to more than one element of the outputs. For a many to many relation, at least one element from inputs is mapped onto one element of the outputs and vice versa. In a one to one relation one element of inputs is mapped onto a unique element in the set of outputs.

Many to one   One to many   Many to many   One to one
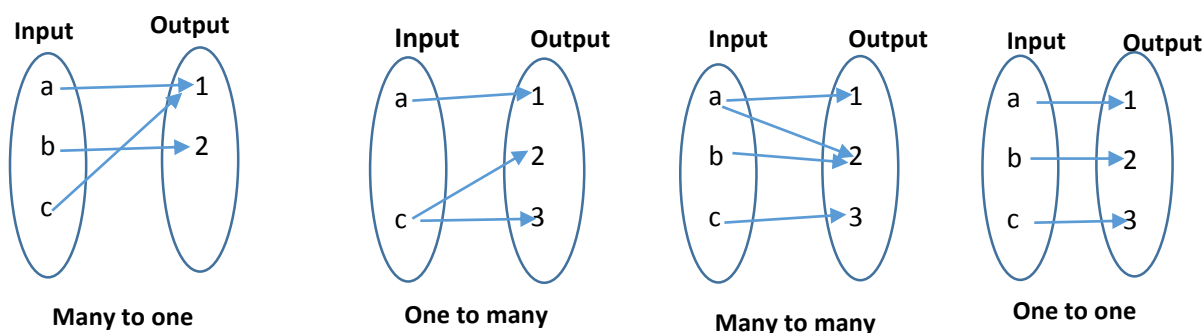
A one to one relation is called a function. In a function every element of the input set has a unique image in the set of outputs. This is usually expressed as

$$y = f(x)$$

where x is some quantity from the set of inputs, which when substituted into the function f, produces the quantity y, which is the set of outputs.

The set of all possible input values of a function is called the domain. The set of all possible output values is called the domain.

A function is most of the time expressed as a mapping between the sets X, Y represented as $f : X \rightarrow Y$. The value of a particular function for a particular value of x, say x = b, is called the **image** of b under f, denoted f(a).

For a function defined by f(x) = 2x – 1, Find f(1) and f(-2).

f(1) means the input is 1 and f(-2) means input is -2. Substitutes these values for x and simplify.

That is: f(1) = 2(1) – 1 = 1   **and**   f(-2) = 2(-2) – 1 = -5

Functions may be defined or undefined for certain value (s). For example, the function f(x) = x + 1 is defined for all values of x. however, the function $f(x) = \frac{3x-1}{x-1}$ is not defined for x = 1. Because when x = 1, the denominator will be zero, which is undefined.

5.1 Types of functions

- Linear function. It is of the type $f(x) = ax + b$, where 1 is the highest power of the variable (x).
- Quadratic function. It is of the type $f(x) = ax^2 + bx + c$, where 2 is the highest power of the variable (x).
- Cubic function: it is of the form $f(x) = ax^3 + bx + cx + d$, where 3 the highest power of the variable(x).

5.2 How to find the Inverse of a function

For the function $f(x) = \frac{2x-3}{x}$, $x \neq 0$, find its inverse.

Solution

Step 1. In the original function, replace $f(x)$ with y.

$$y = \frac{2x-3}{x}$$

Step 2. Make x subject of the formula.

$$\frac{y}{1} = \frac{2x-3}{x} \qquad \text{cross multiply}$$

$$xy = 2x - 3 \qquad \text{collect like terms}$$

$$xy - 2x = -3$$

$$x(y-2) = -3 \qquad \text{divide by } y-2 \text{ on both sides of the equation}$$

$$x = \frac{-3}{y-2}$$

Step 3. Interchange x and y and then replace y with $f^{-1}(x)$.

$$y = \frac{-3}{x-2}$$

Therefore, $\mathbf{f^{-1}(x)} = \frac{-3}{x-2}$

Example 1.

For the function $g(x) = \frac{4x-3}{2x-5}$,

(a) Find $g(-1)$
(b) Find $g^{-1}(1)$

Solution

(a) In the original function, replace x with -1 and simplify

$$g(-1) = \frac{4(-1)-3}{2(-1)-5}$$

$$= \frac{-7}{-7}$$

$$= \underline{\mathbf{1}}$$

(b) First find $g^{-1}(x)$

Step 1. In the original function, replace g(x) with y.

$$y = \frac{4x-3}{2x-5}$$

Step 2. Make x subject of the formula.

$$\frac{y}{1} = \frac{4x-3}{2x-5} \qquad \text{cross multiply}$$

$$y(2x-5) = 4x-3 \qquad \text{expand and \textbf{collect} terms containing x to the left}$$

$$2xy - 5y = 4x - 3$$

$$2xy - 4x = 5y - 3 \qquad \textbf{factorize}$$

$$x(2y-4) = 5y-3 \qquad \textbf{divide by 2y – 4 on both sides of the equation}$$

$$x = \frac{5y-3}{2y-4}$$

Step 3. Interchange x and y and then replace y with $g^{-1}(x)$.

$$y = \frac{5x-3}{2x-4}$$

Therefore, $\mathbf{g^{-1}(x)} = \frac{\mathbf{5x-3}}{\mathbf{2x-4}}$

Step 4: to find $g^{-1}(1)$, replace x with 1 in $g^{-1}(x)$ and then simplify

$$g^{-1}(1) = \frac{5(1)-3}{2(1)-4}$$

$$= \frac{2}{-2}$$

$$= \textbf{-1}$$

Example 1.

Given that $x * y = \frac{x}{y} + \frac{1}{2}$, find the value of $\frac{3}{4} * 2$.

Compare $x * y$ with $\frac{3}{4} * 2$ we see that $x = \frac{3}{4}$ and $y = 2$. To find $\frac{3}{4} * 2$ substitute the values of x and y as shown below

Solution

$$\frac{3}{4} * 2 = \frac{\frac{3}{4}}{2} + \frac{1}{2}$$

$$= (\frac{3}{4} \div 2) + \frac{1}{2}$$

$$= \frac{3}{8} + \frac{1}{2}$$

$$= \frac{7}{8}$$

Example 2

Given that $x * y = xy^2$, find the value of $2 * 3$

Solution

$2 * 3 = 2 \times 3^2$

$= 18$

6. Quadratic functions

This is a function of the type $f(x) = ax^2 + bx + c$ where $a \neq 0$

6.1 Solving quadratic equation

Three methods can be used to solve quadratic equations, by factorization, by completing the square and by using the formula.

(i) **Factorization.** This method is the simplest, however it is not always helpful as they are some quadratic equation which cannot be factorized. When an equation cannot be factorized other methods can be used.

Solve - $3x^2 - 7x - 4 = 0$

First factorize the left hand side. The product is 12, sum is -7 and factors are -3 and – 4

$- 3x^2 – 7x – 4 = -3x^2 – 3x – 4x – 4$

$= (-3x^2 – 3x) – (4x + 4)$

$= - 3x (x + 1) – 4(x + 1)$

$= (-3x – 4) (x + 1)$

Finally equate the factors to 0

$(-3x – 4) (x + 1) = 0$

$-3x – 4 = 0$ or $x + 1 = 0$

$-3x = 4$ $x = -1$

$x = -\dfrac{4}{3}$

(ii) Completing the square

When using this method the coefficient of x^2 should be 1.

Solve $x^2 – 4x – 4 = 0$

add 4 on both sides of the equation

$x^2 – 4x – 4 + 4 = 0 + 4$

$x^2 – 4x = 4$

Make $x^2 – 4x$ into a perfect square. Divide the coefficient in x by 2, square it and add it on both sides of the equation

$x^2 – 4x + (-2)^2 = 4 + (-2)^2$

$(x – 2)^2 = 8$

Now take the square root of each side:

$x – 2 = \pm \sqrt{8}$

$x = 2 \pm \sqrt{8}$

$x = \mathbf{4.83}$ or $x = \mathbf{- 0.83}$

(iii) **Formula method**

For an equation of the form $ax^2 + bx + c = 0$ ($a \neq 0$)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(a) Solve $(2x - 1)(3x - 2) = 3$

Express this equation in the form $ax^2 + bx + c = 0$ by expanding and collecting like terms.

$2x(3x - 2) - 1(3x - 2) = 3$

$6x^2 - 4x - 3x + 2 = 3$

$6x^2 - 7x - 1 = 0$

Comparing $6x^2 - 7x - 1$ with $ax^2 + bx + c$ we get $a = 6$, $b = -7$ and $c = -1$.
Substitute these values in the formula and solve for x.

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-1)}}{2(6)}$$

$$x = \frac{7 \pm \sqrt{49 + 24}}{12}$$

$$x = \frac{7 \pm \sqrt{73}}{12}$$

Therefore, $x = \frac{7 + \sqrt{73}}{12}$ or $x = \frac{7 - \sqrt{73}}{12}$

       x = 1.30   **x = - 0.13**

Note that a quadratic equation has two solutions.

In some cases the variable x may not be used e.g. $m^2 - m + 1$. To solve this equation follow the same procedure above but just replace x with m.

(iv) **Graphical method**

The curve of a quadratic function ($ax^2 + bx + c$) is called a parabola.

When $a > 0$ (e.g. $2x^2 - x + 1$), the parabola has a minimum value at the bottom of the curve and when $a < 0$ (e.g. $-x^2 + 2x + 1$), the graph has a maximum value at the top of the curve.

The minimum value is the value of f (x) at the bottom of the curve. The maximum value is the value of f (x) at the top of the curve. The minimum and maximum values are also called the turning points of the curve.
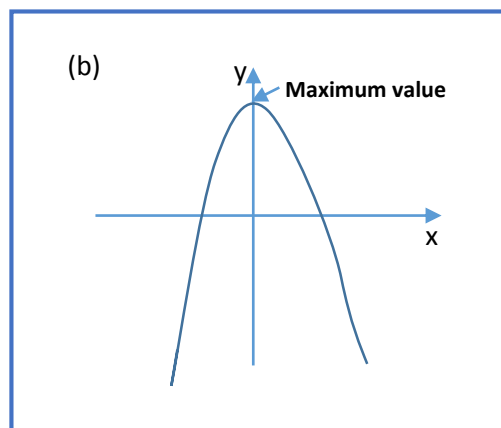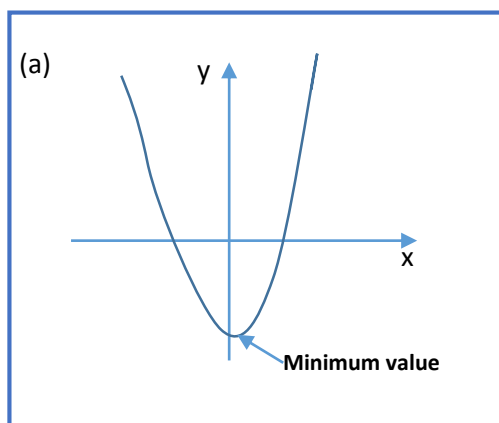
Example

The variable x and y are connected by the equation $y = \frac{1}{5}x(10 - x^2)$.

The table below shows some corresponding values of x and y. the values of y are given correct to one decimal place where necessary.

x	-1	0	1	2	3	3.5	4
y	-1.8	0	1.8	2.4	0.6	-1.6	p

(a) Calculate the value of p.   -4.8
(b) Using a scale of 2cm to represent 1 unit on both axes for $-1 \leq x \leq 5$ and $-5 \leq y \leq 3$, draw the graph of $y = \frac{1}{5}x(10 - x^2)$.
(c) By drawing a tangent a tangent to the curve, estimate the gradient of the curve at the point (1, 1.8).
(d) On the same graph, draw the line whose equation is $5y + 4x = 4$.
(e) Use your graphs to find the solutions of $\frac{1}{5}x(10 - x^2) = -\frac{4}{5}x + \frac{4}{5}$
(f) Estimate the area under the curve between $x = 1$, $x = 3$ and the $y = 0$.

Solution

(a) From the table, the x value corresponding with p is 4. Substitute this value in the equation and solve.

$$p = \frac{1}{5}x(10 - x^2)$$
$$= \frac{1}{5}(4)(10 - 4^2)$$
$$= -4.8$$

(b)
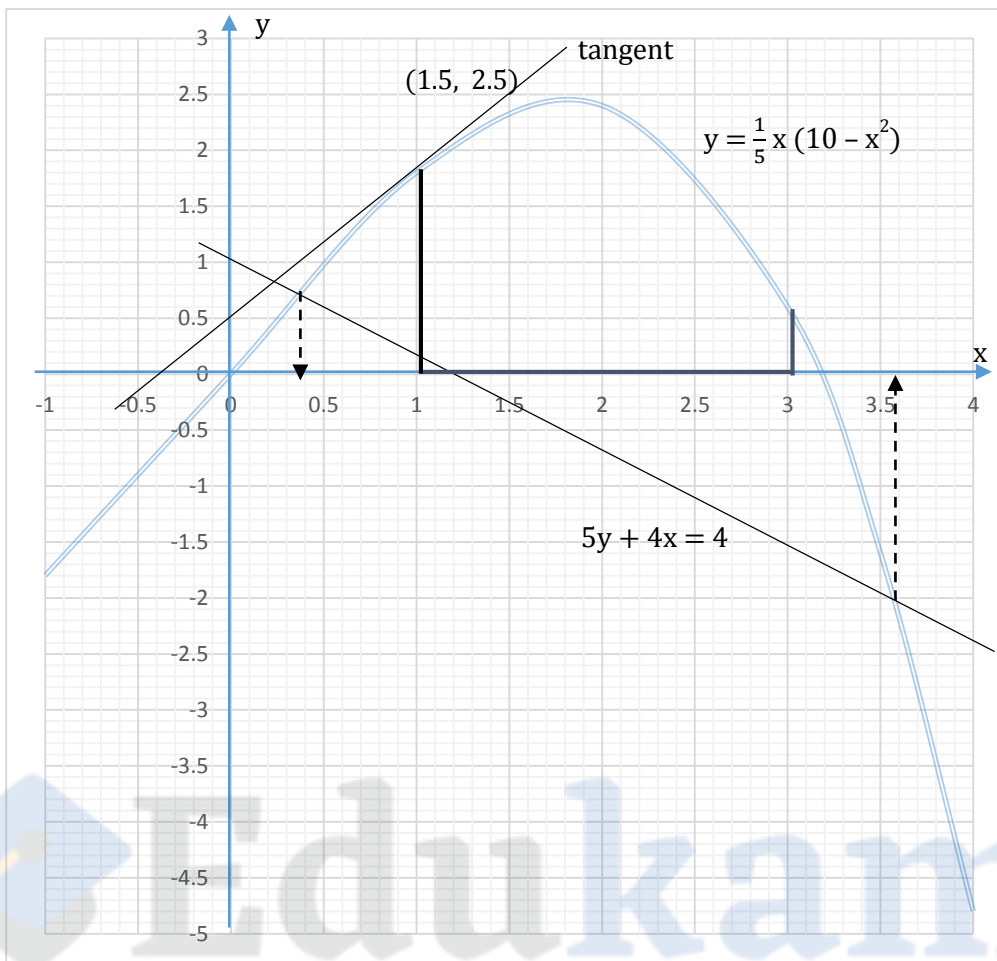
(c) Since the co – ordinates of one of the points lying on the tangent has been given (1, 1.8), find the co – ordinates for any other point that is lying on the tangent, e.g. (1.5, 2.5). here x_1 = 1, y_1 = 1.8 and x_2 = 1.5, y_2 = 2.5, substitute these coordinates in the formula below

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2.5 - 1.8}{1.5 - 1} = \textbf{1.4}$$

(d) To draw the line 5y + 4x = 4, find the co – ordinates of two points using a table of values.

x	1	4
y	0	-2.4

To complete the table, first complete the second column by assuming the value of y = 0, substitute in the equation and solve for the corresponding value of x. e.g. when y = 0, x = 1.
For the third column assume x = 4, substitute in the equation (5y + 4x = 4) and find the corresponding value of y. e.g. when x = 4, y = -2.4. Thus the points are (1, 0) and (4, -2.4), plot them on the graph and join the using a straight line.

(e) the x – values where the curve and the straight line meet are the solutions

x = -0.51 or x = 3.6

(f) Draw the lines satisfying the given condition, here they have been drawn in bold and include the curve. Each box here is 1cm by 1cm = 1cm², to find the area count the number of complete boxes and approximate the area of incomplete boxes then add.
On the graph above the approximate Area = **16 cm²**

7.Variations

7.1 Direct variation

Two variables y and x are said to vary directly when one variable increases the other variable increases by a constant. This is symbolically expressed as y α x, read as y varies directly as x. A direct variation linear equation is of the type   y = kx,   where k is the constant of variation and k ≠ 0.

Example

Given that y is directly proportional to x and that y = 12 when x = 2. Find the value of y when x = 4.

Solution

This is expressed as

$$y = k \, x$$

Substitute for x = 2 and y = 12, then solve for k

$$12 = 2k$$

Divide by 2 on both sides

$$k = 6$$

The general equation then becomes

$$y = 6x$$

Substitute x = 4, in the equation and solve for y.

$$y = 6 \, (4)$$

$$\mathbf{y = 24}$$

A direct variation is called Joint when more than one variable is involved in a direct variation problem. e.g. if P varies jointly with values a and b the equation will be P = kab.

7.2 Inverse variation

Two variables y and x are said to vary inversely when one variable increases the other variable decreases by a constant. this is symbolically expressed as y α $\frac{1}{x}$, read as y varies inversely as x. an inverse variation linear equation is of the type   y = $\frac{k}{x}$,   where k is the constant of variation and k ≠ 0.

Example

Given that y is inversely proportional to x and that y = 4 when x = 2. Find the value of x when y = 4.

Solution

This is expressed as

$$y = \frac{k}{x}$$

Substitute for x = 2 and y = 4, then solve for k

$$4 = \frac{k}{2}$$

Multiply by 2 on both sides

$$k = 8$$

The general equation then becomes

$$y = \frac{8}{x}$$

Substitute y = 4, in the equation and solve for x.

$$4 = \frac{8}{x}$$

Multiply by $\frac{1}{8}$ on both sides of the equation

$$x = \frac{1}{2}$$

Task

y varies directly as Z and inversely as w^2 when y = 6, Z = 2 and w = 3. Find the values of

(a) k
(b) y when z = 6 and w = 9
(c) w when y = 3 and z = 4

8. Matrices

A matrix is an arrangement of elements in rows and columns. For example

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

→ **Row**

↓ **Column**

8.1 Naming of matrices

The name of a matrix is derived from the number of rows and columns it has. That is row by column. For example, a matrix with 1 row and 2 columns is called a 1 X 2 and a matrix with 2 rows and 2 columns is called a 2 X 2.

(i) $\begin{bmatrix} a & b \end{bmatrix}$ 1 x 2 matrix

(ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ 2 x 2 matrix

8.2 Properties of matrices

a. Addition

Add corresponding elements. Addition is commutative. That is matrix A + matrix B = matrix B + matrix A

Let A = $\begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix}$

A + B = $\begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}$ + $\begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix}$

= $\begin{bmatrix} 1+2 & -3+1 \\ 2+0 & 3+7 \end{bmatrix}$

= $\begin{bmatrix} 3 & -2 \\ 2 & 10 \end{bmatrix}$

b. Subtraction

Subtract corresponding elements. Subtraction is not commutative. That is matrix A – matrix B is not the same as matrix B – matrix A.

A - B = $\begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}$ - $\begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix}$

= $\begin{bmatrix} 1-2 & -3-1 \\ 2-0 & 3-7 \end{bmatrix}$

= $\begin{bmatrix} -1 & -4 \\ 2 & -4 \end{bmatrix}$

c. **Multiplication.**

Multiply the row by the column

$$AB = \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} (1 \times 2) + (-3 \times 0) & (1 \times 1) + (-3 \times 7) \\ (2 \times 2) + (3 \times 0) & (2 \times 1) + (3 \times 7) \end{pmatrix}$$

$$= \begin{pmatrix} 2 + 0 & 1 + (-21) \\ 4 + 0 & 2 + 21 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{2} & \mathbf{-20} \\ \mathbf{4} & \mathbf{23} \end{pmatrix}$$

d. **Multiplying a matrix by a scalar (a constant)**

Multiply each element of the matrix by a scalar (constant) to get the new matrix.

$$2A = 2 \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{2} & \mathbf{-6} \\ \mathbf{4} & \mathbf{6} \end{pmatrix}$$

e. **Equal matrices**

Two or more matrices are equal if and only if they contain the same elements arranged in the same order.

$$\text{Thus, } A \neq B : \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 0 & 7 \end{pmatrix}$$

f. **Transpose of a matrix**

A transpose of a matrix is one in which the rows of the original matrix have been reversed to columns and the rows to columns.

$$A = \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix} \qquad\qquad C = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$$

$$A^{T} = \begin{pmatrix} 1 & 2 \\ -3 & 3 \end{pmatrix} \qquad\qquad C^{T} = \begin{pmatrix} 1 & 5 & 7 \end{pmatrix}$$

8.3 Determinant of a 2 x 2 matrix

Let A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

To find the determinant

- ✓ Multiply the elements in the major diagonal
- ✓ Multiply the elements in the minor diagonal
- ✓ Finally, determinant = product of major diagonal – product of minor diagonal

$$\boxed{\det A = ad - bc}$$

8.4 Singular matrix

A singular matrix is one which has the determinant of zero (0).

8.5 Inverse of a 2 x 2 matrix

Let A = $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

Step 1: find the determinant of the matrix.

$$\det A = (2 \times 1) - (1 \times 0) = \mathbf{2}$$

Step 2: in the original matrix, interchange the elements in the major diagonal and change the signs of the elements in the minor diagonal to get the new matrix B.

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

Step 3: *multiply* $\frac{1}{\det A}$ *with the new matrix* B.

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

Step 4. The product in step 3 is the inverse matrix. The superscript $^{-1}$ is used to denote the inverse.

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

8.6 Solving simultaneous equations in two variables using a matrix methods.

$$2x + y = 4$$

$$x - 2y = 2$$

Step 1: convert the equations into matrices. Form a 2 x2 matrix using the coefficients such that those of equation (i) form the upper (1st) row and those of equation (ii) form the 2nd row. Next to this matrix form a 2 x 1 matrix of the variables such that it is x above y. finally, form another 2 x 1 matrix of constants after the equal sign.

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Step 2: find the inverse of the 2 x 2 matrix

$$\text{Let A} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\det A = (2 \times -2) - (1 \times 1)$$

$$= -4 - 1 = \mathbf{-5}$$

$$\text{Therefore } A^{-1} = -\frac{1}{5} \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}$$

Step 3: multiply by the inverse on both sides of the equation

$$-\frac{1}{5} \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$-\frac{1}{5} \begin{pmatrix} (-2 \times 2) + (-1 \times 1) & (-2 \times 1) + (-1 \times -2) \\ (-1 \times 2) + (2 \times 1) & (-1 \times 1) + (2 \times -2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} (-2 \times 4) + (-1 \times 2) \\ (-1 \times 4) + (2 \times 2) \end{pmatrix}$$

$$-\frac{1}{5} \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5} \begin{pmatrix} -10 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \textbf{these are equal matrices, therefore}$$

$$x = 2 \quad \text{and} \quad y = 0$$

It should be noted that when you multiply a matrix by its inverse you will get an identity matrix.

Thus, $\boxed{\text{AA}^{-1} = \text{identity matrix}}$

For a 2 x 2 matrix the identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

When solving simultaneous equations in two variables using the matrix method, you can save yourself some trouble of expanding the left hand side by using identity matrix property. In short state the property and jump to the step involving the identity matrix.

Task 1

If $P = \begin{pmatrix} 2 & 0 \\ 6 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, find

(i) PQ,
(ii) the values of a and b, given that PQ = P – Q

Solution

(i) $PQ = \begin{pmatrix} 2 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} = \begin{pmatrix} 2a + 0 & 0 + 0 \\ 6a + 1 & 0 + b \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 6a + 1 & b \end{pmatrix}$

(ii) PQ = P – Q

$\begin{pmatrix} 2a & 0 \\ 6a + 1 & b \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 6 & 1 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$

$\begin{pmatrix} 2a & 0 \\ 6a + 1 & b \end{pmatrix} = \begin{pmatrix} 2 - a & 0 \\ 5 & 1 - b \end{pmatrix}$

These two matrices are equal, equate the elements containing letter of choice to their corresponding elements and solve

$2a = 2 - a \qquad\qquad b = 1 - b$

$3a = 2 \qquad\qquad\qquad 2b = 1$

$a = \dfrac{2}{3} \qquad\qquad\qquad b = \dfrac{1}{2}$

Task

Given that matrix $A = \begin{pmatrix} 1 & x \\ -1 & 2 \end{pmatrix}$,

(i) write an expression in terms of x, for the determinant of A.
(ii) find the value of x, given that the determinant of A is 5,
(iii) write A^{-1}.

9. Vectors in two dimensions

A vector is a quantity that has magnitude (size) and direction.

9.1 Vector representation

Vectors can be represented as a directed line segment (as arrows).

To represent the vector displacement from point P to point Q, the following notations can be used:

$$\overrightarrow{PQ} \qquad or \qquad \underline{PQ}$$

The vector $\underline{a}$ is called a position vector because it is fixed to the origin. When moving against the line of segment the sign of the vector changes e.g - $\underline{a}$. The vector – $\underline{a}$ has same magnitude as $\underline{a}$ but is in the opposite direction.

The magnitude of vector $\underline{a}$ is represented as $|\underline{a}|$.

9.2 Addition and subtraction of vectors

The triangle rule is used to add vectors. For example $\underline{a}$ and $\underline{b}$ are two sides of a triangle in order and $\underline{c} = \underline{a} + \underline{b}$ is the resultant of $\underline{a}$ and $\underline{b}$.

$$\underline{c} = \underline{a} + \underline{b}$$

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$$
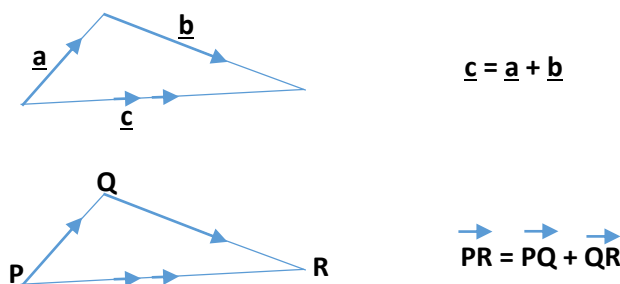
9.3 Matrix representation of vectors

Vector matrix – is a 2 x 1 matrix of the type $\begin{bmatrix} x \\ y \end{bmatrix}$ read as x above y.
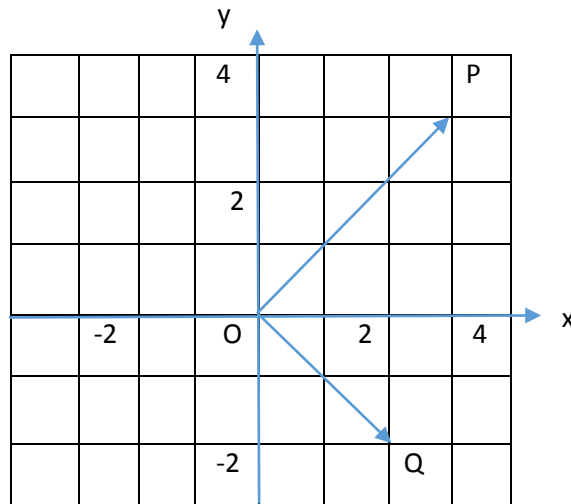
$$\overrightarrow{OP} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad \overrightarrow{OQ} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \qquad \overrightarrow{PO} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \qquad \overrightarrow{QO} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

9.4 Addition and subtraction vector matrix

Follow the triangle rule and rule of matrices. From the previous example

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$

$$= \begin{bmatrix} -3 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{-1} \\ \mathbf{-5} \end{bmatrix}$$

9.5 Magnitude of a vector matrix

$|\text{Magnitude}| = \sqrt{x^2 + y^2}$

For example. Find the magnitude of $\overrightarrow{PQ} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$

$$|\overrightarrow{PQ}| = \sqrt{(-1)^2 + (-5)^2}$$

$$= \sqrt{\mathbf{26}} \ \mathbf{units}$$

Example 1

The diagram below is a trapezium OABC. M is the midpoint of AB, OM and CA meet at X.
$\vec{OA} = 4p$, $\vec{OC} = 2p$.
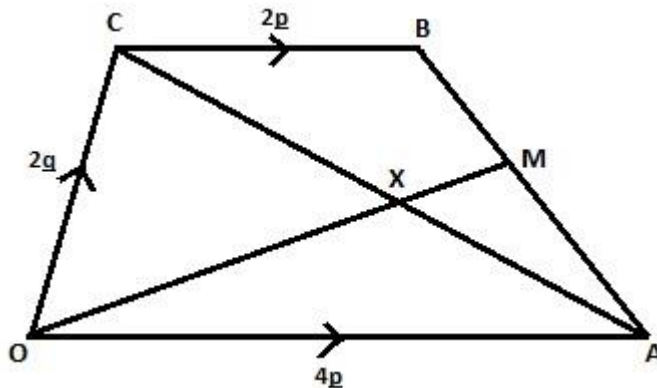
(i)   Express as simply as possible, in terms of p and/or q

(a) $\vec{CA}$,

(b) $\vec{BA}$,

(c) $\vec{OM}$.

(ii)   Given that CX = hCA, express $\vec{CX}$ in terms of p, q and h.

(iii)   Hence, show that $\vec{OX} = 4hp + 2(1 – h)q$

Solution

(i)   (a) $\vec{CA} = \vec{CO} + \vec{OA}$

        $= -2q + 4p$ note that $\vec{CO} = -\vec{OC} = -2q$ i.e moving against the line segment

        $= 4p – 2q$ (multiply by – ve when moving opposite in direction)

        (b) $\vec{BA} = \vec{BC} + \vec{CA}$

        $= -2p + 4p – 2q$ note that $\vec{BC} = -\vec{CB} = -2p$

        $= 2p – 2q$

        (c) $\vec{OM} = \vec{OA} + \vec{AM}$ $\vec{AM} = \frac{1}{2}\vec{AB}$

        $= 4p + q – p$ $= \frac{1}{2}(2q – 2p)$

        $= 3p + q$ $= q – p$

(ii) $\vec{CX} = h\vec{CA}$

        $= h(4p – 2q)$

        $= 4hp – 2hq$

(iii) $\vec{OX} = \vec{OC} + \vec{CX}$

        $= 2q + 4hp – 2hq$ collect like terms

        $= 4hp + 2q – 2hq$ factorize

        $= 4hp + 2(1 – h)q$ hence shown

Example 2

In the diagram below, OT : TC = 2 : 1, $\vec{OA} = 2\mathbf{a}$, $\vec{OB} = \mathbf{b}$ and $\vec{BC} = \mathbf{a}$. express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.
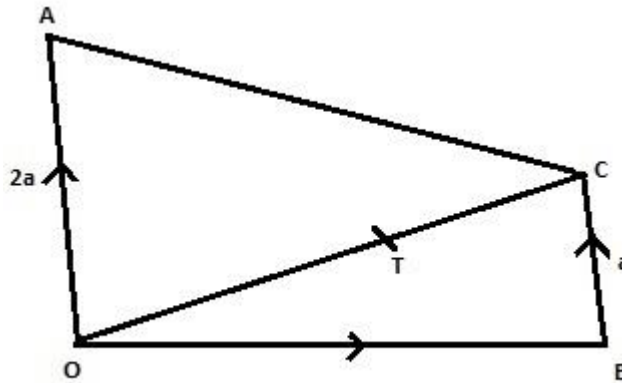
(i) $\vec{OC}$

(ii) $\vec{TC}$

(iii) $\vec{BT}$

(iv) $\vec{TA}$

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Solution

(i) $\vec{OC} = \vec{OB} + \vec{BC}$

        $= \mathbf{b} + \mathbf{a}$

        $= \mathbf{a} + \mathbf{b}$

(ii) $\vec{TC} = \frac{1}{3}\vec{OC}$

        $= \frac{1}{3}(\mathbf{a} + \mathbf{b})$

(iii) $\vec{BT} = \vec{BC} + \vec{CT}$

        $= \mathbf{a} + \frac{1}{3}(\mathbf{a} + \mathbf{b})$ expand, collect like terms and simplify

        $= \frac{4}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

(iv) $\vec{TA} = \vec{TO} + \vec{OA}$ $\vec{TO} = -\frac{2}{3}\vec{OC} = -\frac{2}{3}(\mathbf{a} + \mathbf{b})$

        $= -\frac{2}{3}(\mathbf{a} + \mathbf{b}) + 2\mathbf{a}$ expand

        $= -\frac{2}{3}\mathbf{a} - \frac{2}{3}\mathbf{b} + 2\mathbf{a}$ collect like terms and manipulate like algebraic fraction

        $= 2\mathbf{a} - \frac{2}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$

        $= \frac{4}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$

</div>

Example 3

Given that $\underline{a} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$, find $|\underline{a}|$.

Solution

$|\underline{a}| = \sqrt{(-8)^2 + (6)^2}$

$= \sqrt{100}$

= 10 units

Example 4

The diagram below shows a trapezium in which $\overrightarrow{OA} = a$, $\overrightarrow{OC} = c$ and $\overrightarrow{CB}$ is parallel to $\overrightarrow{OA}$. $\overrightarrow{CB}$ is twice $\overrightarrow{OA}$, the points D and E are midpoints of AB and CB.
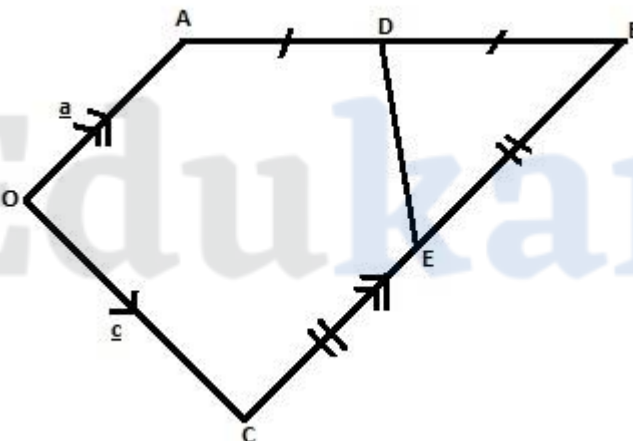
Express in terms of a and/or b the following vectors

(i) $\overrightarrow{CA}$,

(ii) $\overrightarrow{AB}$,

(iii) $\overrightarrow{EB}$,

(iv) $\overrightarrow{OD}$,

(v) $\overrightarrow{BO}$.

Solution

(i) $\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA}$

   $= -\underline{c} + \underline{a}$ note CO = -OC = -$\underline{c}$

   $= \underline{a} - \underline{c}$

(ii) $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$

   $= \underline{c} - \underline{a} + 2\underline{a}$ note AC = - CA = - (a – c) = c - a

   $= \underline{a} + \underline{c}$

(iii) $\overrightarrow{EB} = \frac{1}{2}\overrightarrow{CB}$ remember that E is the midpoint

   $= \frac{1}{2}(2\underline{a})$

   $= \underline{a}$

(iv) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

   $= \underline{a} + \frac{1}{2}(\underline{a} - \underline{c})$ expand, collect like terms and manipulate like

   $= \underline{a} + \frac{1}{2}\underline{a} - \frac{1}{2}\underline{c}$ normal algebraic fraction

   $= \frac{3}{2}\underline{a} - \frac{1}{2}\underline{c}$

(v) $\overrightarrow{BO} = \overrightarrow{BC} + \overrightarrow{CO}$

   $= -2\underline{a} + (-\underline{c})$ note that BC and CO are negative (moving against line of segment)

   $= -2\underline{a} - \underline{c}$

Task

In the diagram below, $\overrightarrow{OP} = 2\underline{p}$ and $\overrightarrow{OQ} = 2\underline{q}$. X is a point on PQ such that $\frac{PX}{PQ} = \frac{2}{3}$.
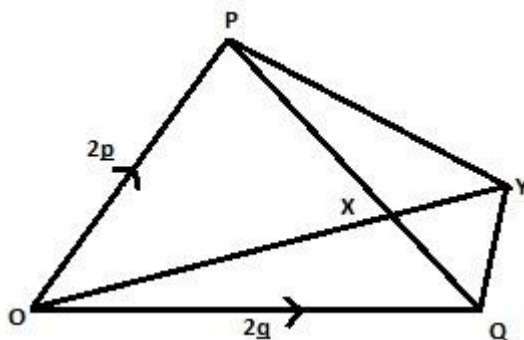
(i)   Express in terms of $\underline{p}$ and/or $\underline{q}$
         (a) $\overrightarrow{PQ}$,
         (b) $\overrightarrow{QX}$.

(ii)   Given also that $\overrightarrow{OY} = h\overrightarrow{OX}$, express $\overrightarrow{OY}$ in terms of $\underline{p}$, $\underline{q}$ and h.

10.   Co – ordinate geometry

Here we will deal with description of point(s) using co – ordinates on the xoy plane. We will also look at the properties of a straight line, how to calculate the gradient, midpoint, and length (distance).

10.1 Description of points on the xoy plane

A point can be described by stating its position on the x – axis and y – axis, (x, y). (x, y) is called Cartesian coordinates of the point.
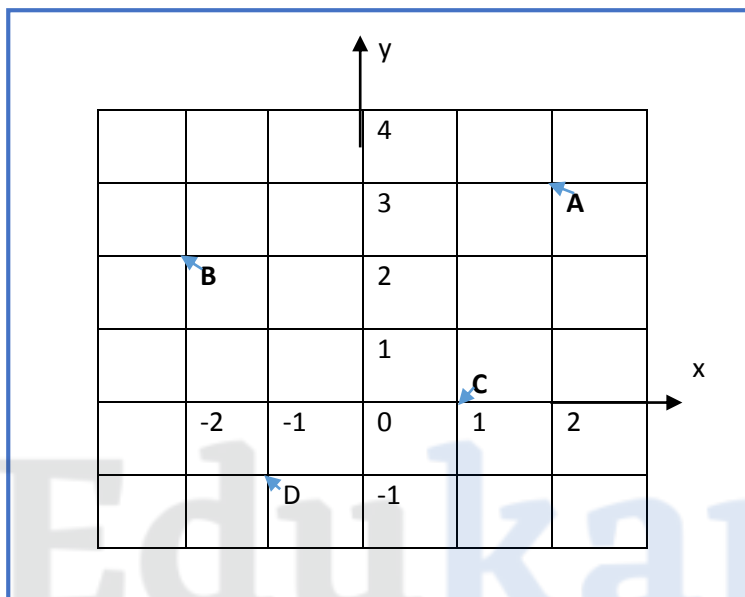
Locate the co – ordinates of the points: A, B and C on the Cartesian plane above.

Locate the position of the point on the x – axis and in the y – axis. Write the coordinate in the form (x, y).

A (2, 3)   B (-2, 2)   C (1, 0)   D (-1, -1)

10.2 Gradient

Gradient refers to how steep a line is. The letter m will be used in this book to represent gradient. **Two point gradient formula**, (x_1, y_1) and (x_2, y_2)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This equation can be used to find the gradient of a straight line given the co –ordinates of two points lying on the line.

Find the gradient of the line AB, given A (2, 3) and B (-2, 2)

Here, take the coordinates of A and name them x_1 and y_1. That is $x_1 = 2$ and $y_1 = 3$ and coordinates of B as x_2 and y_2. That is $x_2 = -2$ and $y_2 = 2$. Substitute these values in the formula for gradient as simplify.

$$m = \frac{2 - 3}{-2 - 2} = \frac{1}{4}$$

The y – intercept form

$$y = mx + c$$

, **where m is the gradient and c is the value where the line intersects the y – axis.**

This equation can be used to find the gradient of a line given its equation. Simply make y subject of the formula like in the equation above. The co – efficient of x is the gradient.

Find the gradients of the following lines:

(a) $2x + y = 3$
Make y subject (form an equation of the type $y = mx + c$)
$y = -2x + 3$
Therefore, m = **-2**

(b) $2y + x - 5 = 0$
$2y = -x + 5$
$y = -\frac{1}{2}x + 5$
Therefore, $m = -\frac{1}{2}$

(c) $1 - y = x$
$1 - x = y$ which is the same as
$y = -x + 1$
Therefore, m = **-1**

Gradient of parallel and perpendicular lines

Let Line 1 (L1) have gradient m_1 and Line 2 (L2) gradient m_2

If L1 and L2 are parallel, then $m_1 = m_2$

If L1 and L2 are perpendicular, then $m_1 m_2 = -1$

10.3 The midpoint

$$midpoint = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

Find the midpoint of the points P (1, 3) and Q (3, 3)

$$midpoint = (\frac{1 + 3}{2}, \frac{3 + 3}{2})$$

$$= (\frac{4}{2}, \frac{6}{2})$$

$$= (2, 3)$$

10.4 Equation of a line

The equation of a straight is given by

$$y - y_1 = m (x - x_1)$$

To find the equation of a line, find the gradient, if two points are given just pick the co – ordinates for one point, name them x_1 and y_1 then substitute in the equation above and simply.

The points A and B are (4, 0) and (0, 8) respectively. Find the equation of AB

Find the gradient of AB first. $X_1= 4$, $y_1= 0$, $x_2 = 0$ and $y_2 = 8$

$$m = \frac{8 - 0}{0 - 4} = \frac{8}{-4} = -2$$

Substitute the values of the gradient and one of the points in the equation

$$y - y_1 = m (x - x_1)$$

$$y - 0 = -2 (x - 4)$$

$$y = -2x + 8$$

$$y + 2x = 8$$

10.5 Distance between two points

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The points A and B are (4, 0) and (0, 8) respectively. Find the distance between of AB

$X_1= 4$, $y_1= 0$, $x_2 = 0$ and $y_2 = 8$

Substitute these values into the formula

$$\text{Distance} = \sqrt{(0 - 4)^2 + (8 - 0)^2}$$

$$= \sqrt{16 + 64}$$

$$= \sqrt{80}$$

$$= 8.94 \text{ units}$$

Task

    (a) Find the equation of line l passing through a point (0, 5) whose gradient is 3.

    (b) if B is a point (6, -3) and C is (-2, 1), find

        (i)   the gradient of line BC

        (ii)   the equation of line BC

11. Linear programming

Linear programming is used to solve maximization and minimization problems of linear functions subject to constraints. These constraints may be inequalities or equalities.

11. 1 Plotting inequalities

Just like linear equations, linear inequalities can be plotted on a graph. Linear inequalities containing > or < symbols are shown as broken lines while those containing ≥ or ≤ symbols are shown as solid lines. After inequalities have been plotted, the unwanted region is shaded. This leaves a region bound by the inequalities called the feasible region. This is the region where all solutions lie.

Example 1

Plot the inequalities below and shade the unwanted region. Indicate clearly the unshaded region by labelling it F.

$x < 4$

$x + y \geq 4$

$y \leq x$

Solution

Firstly, convert the inequalities into equations and find two pairs of coordinates for each inequality using a box of values. Remember you need two points to draw a straight line.

For x < 4

$x = 4$ this is a vertical line that cuts the x – axis at 4, hence no need of finding coordinates

For x + y ≥ 5

$x + y = 5.$

To fill the first column assume x = 0, substitute in the equation and solve for y. for the second column assume y = 0 and solve for x.

x	0	5
y	5	0

Thus we have the coordinates (0, 5) and (5, 0)

For y ≤ x

$y = x$

To fill the first column assume x = 0, substitute in the equation and solve for y. for the second column assume y = 4 and solve for x.

x	0	5
y	0	5

Thus we have the coordinates (0, 0) and (5, 5)
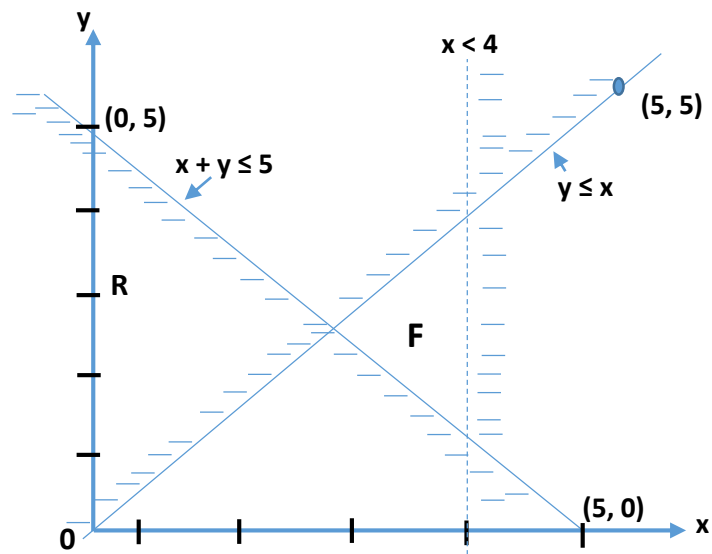
Note that x < 4 should be a broken line and the region the x > 4 is shaded because it is unwanted. The other inequalities are solid and their unwanted regions shaded. The wanted region should be bound by these inequalities.

Example 2

The diagram below shows the region R bounded by three lines. Write the three inequalities describing the unshaded region marked R.
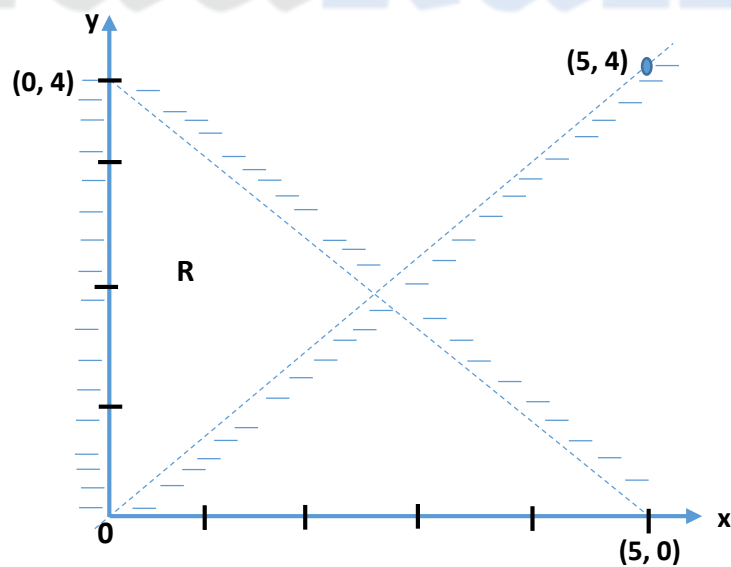
Solution

First identify the lines making the boundary of region R. these are

1) these lines are the y – axis,
2) the line with coordinates (0, 4) and (5,0)
3) the line with coordinates (0, 0) and (5, 4)

Now find the equations of these lines and change them into inequalities depending on where the wanted region is.

1) For the y – axis is vertical line that cuts the x – axis at zero, therefore its equation is
$$x = 0$$
The region R is on the side where x is greater than 0, hence the inequality is
$$x \geq 0$$
The $\geq$ sign has been used because the line is solid.

2) For the line with coordinates (0, 4) and (5, 0), use these coordinates to find the equation like we did in the previous chapter on coordinate geometry.

For the (0, 4) and (5, 0) let $X_1 = 0$, $y_1 = 4$, $x_2 = 5$ and $y_2 = 0$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{0 - 4}{5 - 0} = \frac{-4}{5} = -\frac{4}{5}$$

Using the formula for the equation of a straight line substitute for gradient and coordinates and simplify
$$y - y_1 = m (x - x_1)$$
$$y - 4 = -\frac{4}{5} (x - 0)$$
$$5(y - 4) = -4 (x - 0)$$
$$5y - 20 = -4x$$
$$5y + 2x = 20$$
Now find the inequality sign. The region R is below this line hence the $<$ sign will be used. Also because the line is broken.
$$5y + 2x < 20$$

3) For the (0, 0) and (5, 4) let $X_1 = 0$, $y_1 = 0$, $x_2 = 5$ and $y_2 = 4$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{4 - 0}{5 - 0} = \frac{4}{5} = \frac{4}{5}$$

Using the formula for the equation of a straight line substitute for gradient and coordinates and simplify
$$y - y_1 = m (x - x_1)$$
$$y - 0 = \frac{4}{5} (x - 0)$$
$$5(y - 0) = 4 (x - 0)$$
$$5y = 4x$$

Now find the inequality sign. The region R is above this line hence the $>$ sign will be used. Also because the line is broken.
$$5y > 4x$$

11.2 Formulating inequalities from given information

Important concepts

- At least means ≥
- Not more than ≤
- a total of ≤
- More than means >
- Less than means <
- The Constraints x > 0 and y > 0 (non – negativity constraints) are should always be applied
- The function to be maximized or minimized is called the objective function

Example

Mr. Simukadi intends to buy a total of 500 scratch cards from Unitel and Bitel. He decides to buy Bitel scratch cards which must be at least one third of Unitel scratch cards. He wants to buy at least 150 Bitel scratch cards and not more than 300 Unitel scratch cards. Let x be the number of Unitel scratch cards and y be the number of Bitel scratch cards.

(a) Write down four inequalities which satisfy the above conditions.   [4]
(b) Show these inequalities on a graph paper using 2cm to represent 100 scratch cards on both axes.   [4]
(c) The dealer makes a profit of K500 on each Unitel scratch card and K150 on each Bitel scratch card sold.
   (i)   Find the number of scratch cards of each type he must buy in order to maximize his profit.   [2]
   (ii)   Calculate the maximum profit.   [2]

Solutions

(a) First identify the sentences that have conditions and then form the inequalities.
   (i)   Mr. Simukadi intends to buy **a total of** 500 scratch cards from Unitel and Bitel.
$$x + y \leq 500$$
   (iii)   He decides to buy **Bitel** scratch cards which must be **at least** one third of Unitel scratch cards.
$$y \geq \frac{1}{3}x$$
   (iv)   He wants to buy **at least** 150 Bitel scratch cards.
$$y \geq 150$$
   (v)   He wants to buy **not more than** 300 Unitel scratch cards.
$$x \leq 300$$

(b) nn
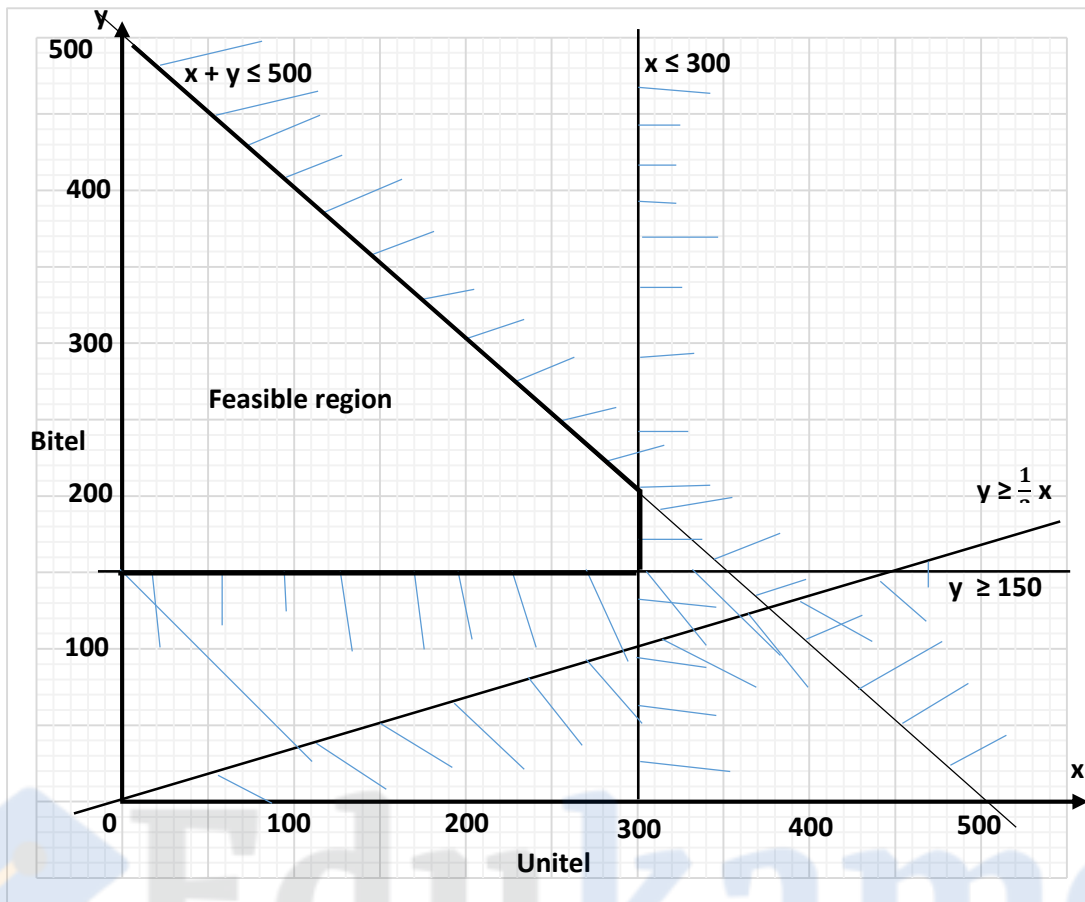
(c) First formulate the objective function, in this case it is for profit

Profit = 500x + 200y

(i) Now pick the coordinates at the corners of the feasible region, and choose one such that when substituted in the objective function gives maximum profit.
The points at the corners are (0, 500), (0, 150), (300, 150) and (300, 200). Here (300, 200) gives the maximum value when substituted in the equation.
Hence to maximize profit, he must sell 300 Unitel scratch cards and 200 Bitel scratch cards.

(ii) Profit = 500x + 200y
= 500(300) + 200(200)
= K190 000

Task

A small scale farmer wishes to keep sheep and goats. Let x represent the number of sheep and y represent the number of goats.

- (i) write the inequalities which represent each of the following conditions:
  - (a) the number of sheep should not be more than 4. [1]
  - (b) a goat feeds on 4kg of food while a sheep feeds on 2kg of food per day. The total amount of food should be at least 8kg per day. [2]
  - (c) the number of sheep should be more than the number of goats. [1]
- (ii) using a scale of 2cm to represent 1 unit on both axes, draw the x and y axes for $0 \leq x \leq 5$ and $0 \leq y \leq 5$ and shade the unwanted region to indicate clearly the region where the solution of the inequalities lie.

12. Angle geometry

An angle is formed when two lines meet. It can be measured by a protractor in degrees. Radian is another unit for measuring angles, however the degree will be used throughout this book.
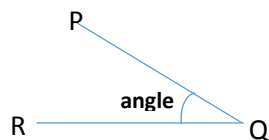
12.1 Types of angles

- An angle that is less than 90° is called an acute angle.

- An angle that is exactly 90° is a right angle.
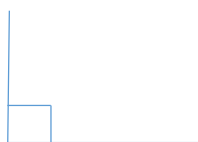
- An angle that is more than 90° but less than 180° is an obtuse angle.

- An angle that is exactly 180° is called a straight angle.

- An angle that is more than 180° but less than 360° is a reflex angle.

12.2 Properties of angles and straight lines

- The angle for a complete revolution is 360°. That is when a point is rotated once such that it returns to its original position, it makes an angle of 360°.
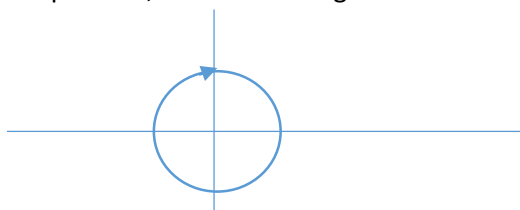
- The total angle on a straight line is 180°.

180°

- For straight lines that cross each other, vertically opposite angles are equal. That is when the lines make an X – shape.

p
s   r
q
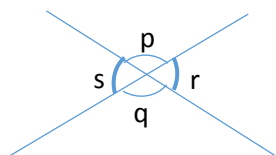
Angle P = angle q
Angle s = angle r

Angle p + Angle r = 180°
(supplementary angles)

- Alternate angles between parallel lines are equal. The Z- property.

n

m
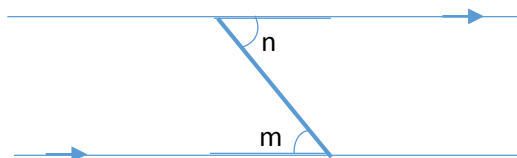
Angle m = angle n

- If parallel lines are cut by a straight line, the corresponding angles are equal.

m
p
o   n

d   a
c   b
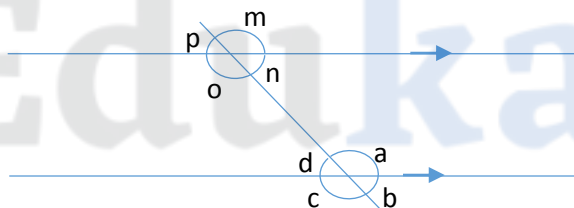
Angle m = Angle a
Angle n = Angle b
Angle o = Angle c
Angle p = Angle d

- Interior angles between parallel lines add up to 180°.

p

q
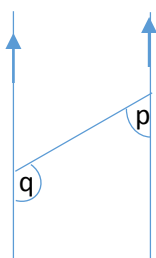
Angle p = Angle q

12.3 Line and rotational symmetry

Shapes have line symmetry if a mirror could be placed so that one side is an exact reflection of the other. The imaginary 'mirror lines' are shown by dotted lines.
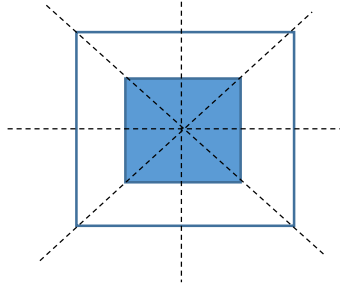
This shape has 4 lines of symmetry

An object has rotational symmetry if it can be rotated about a point so that it fits on top of itself without completing a full turn.
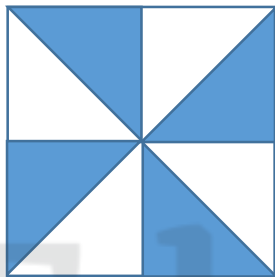
In a complete turn the shape above fits on top of itself four times. It has rotational symmetry of order 4.

Example. Find the number of lines of symmetry and the order of the rotational symmetry of the figure below.
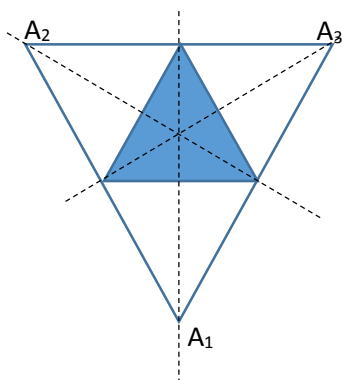
This figure has 3 lines of symmetry.

It has rotational order of 3 because the point marked A_1 could be rotated to A_2 then to A_3 and it fits exactly over the original shape at each of the points.

12.4 Polygons

A polygon is a closed figure that has at least 3 sides. Each side is joined to two other sides at its end points called vertices. Polygons may be regular or irregular. Regular polygons have well defined shapes when compared with irregular polygons. Examples of regular polygons include triangle, quadrilateral and pentagon.

For a polygon of n – sides, it can be divided into n – 2 triangles. Also, the sum of angle measures of a polygon is (n – 2) 180°

A regular polygon is a polygon with all sides congruent and all angles congruent. Since all angles of a regular n –sided polygon are congruent and since its sum of angle measures is $(n-2)180°$, the degrees measure of each interior angle is found as follows:

$$\textbf{Interior Angle} = \frac{(n-2)180°}{n}$$

12.5 Rotational and line Symmetry of Regular Polygon

For a regular polygon of n – sides, it has n – fold rotational symmetry and the centre of rotation is the centre of the polygon.

All regular polygons have line symmetry. A polygon of n – sides have n – lines of symmetry. That is a polygon of 6 sides (hexagon) has six lines of symmetry.

12.6 Similarity and congruency

Two shapes are **congruent** if they have the same shape and size: that is, their corresponding sides have the same shape and the same length. Their corresponding angles should also be the same. The symbol for congruency is ≅. These two triangles are congruent.
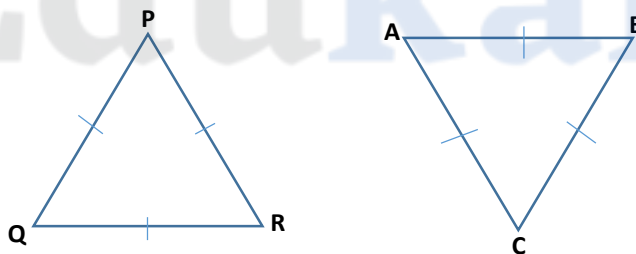
Shapes are similar if they have the same shape but the sizes may be different. That is, if two shapes are similar, then the ratio of the lengths of any two sides of one shape is equal to the ratio of the lengths of the corresponding sides of the other shape. The symbol for similarity is ~.

Consider these two rectangles below. If these are similar, then the ratio of the lengths of any two sides of rectangle ABCD is equal to the ratio of the lengths of the corresponding sides of rectangle A'B'C'D'.

That is $\dfrac{AB}{BC} = \dfrac{A'B'}{B'C'}$
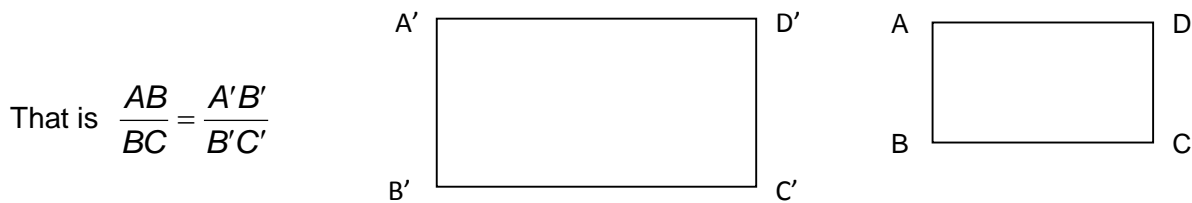
Example 1

In the diagram, ABCD is a quadrilateral in which $\stackrel{\wedge}{ADC} = 140°$, $\stackrel{\wedge}{ABC} = 55°$ and the point Y on AB is such that YD is parallel to BC.
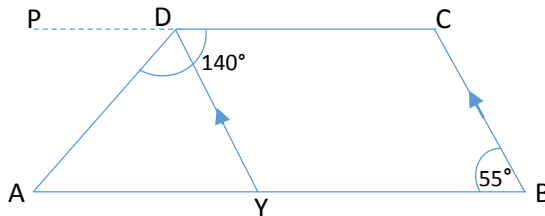
Calculate

(a) $\stackrel{\wedge}{BYD}$

(b) $\stackrel{\wedge}{BAD}$

Solution

(a) $\stackrel{\wedge}{BYD} = 180° - \stackrel{\wedge}{BYD}$ note $\stackrel{\wedge}{ABC} = \stackrel{\wedge}{AYD} = 55°$
   $= 180° - 55°$
   $= 125°$

(b) Extend line CD to P as shown by dotted line in the diagram above. now use the Z property (PDAY is making Z)

$\stackrel{\wedge}{PDA} = 180° - \stackrel{\wedge}{ADC}$
   $= 180° - 140°$
   $= 40°$

$\stackrel{\wedge}{BAD} = \stackrel{\wedge}{PDA} = 40°$

Task

In the diagram below, PQR is parallel to TS, QUT is a straight line, $\stackrel{\wedge}{QRU} = 30°$, $\stackrel{\wedge}{TSU} = 20°$ and $\stackrel{\wedge}{SUT} = 97°$.
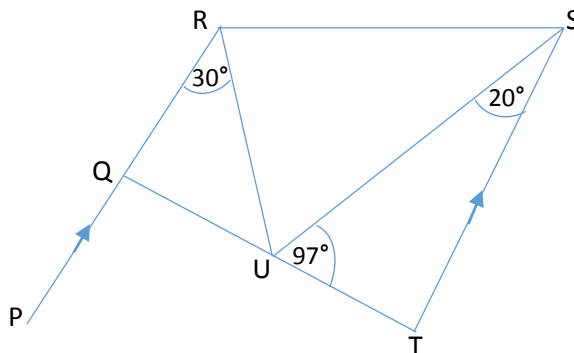
Calculate

(i) $\stackrel{\wedge}{PQT}$

(ii) $\stackrel{\wedge}{RUS}$

Rectangle and Rhombus
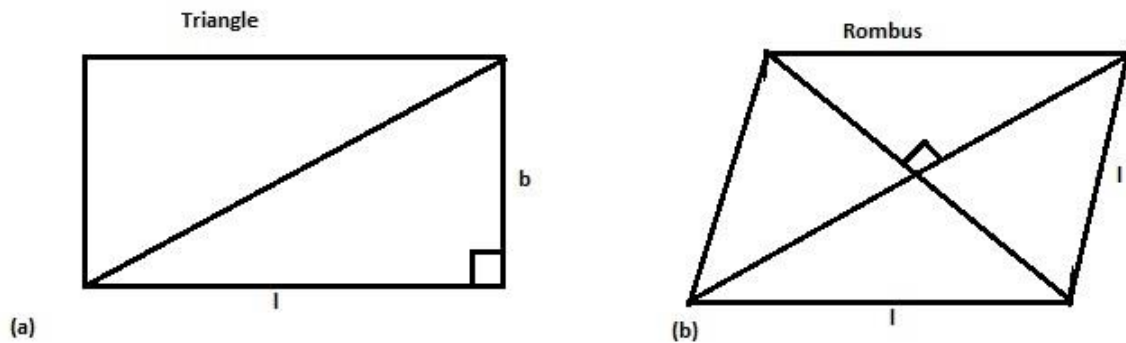
The diagonal of a rectangle divides it into two equal right angled triangle since each interior angle of a rectangle is 90°. In this case the diagonal forms the hypotenuse. To find the diagonal or length or breadth the Pythagoras theorem can be used.

The diagonal of a square also divides it into two equal isosceles triangles.

A rhombus is a square which has been slant. The diagonals of a rhombus intersect half way (at centre) at an angle of 90°. the length of the rhombus forms the hypotenuse.

Example

The diagonals of a rhombus are 20cm and 16cm long. Calculate the length of the rhombus, giving your answer correct to 2 decimal points.

Solution

We use the Pythagoras theorem, where l the length of the rhombus is the hypotenuse, and we halve the diagonals to get the opposite and adjacent sides. Opposite = 10cm and adjacent = 8cm

$$l^2 = (10cm)^2 + (8cm)^2$$

$$= \sqrt{164cm^2}$$

$$= \mathbf{12.81cm}$$

Example 2

A rectangle has a length of 4cm and a breadth of 3cm, calculate the length of the diagonal.

Solution

Let d represent the diagonal.

$$d^2 = (4cm)^2 + (3cm)^2$$

$$= \sqrt{25cm^2}$$

$$= \mathbf{5\ cm}$$

13.   Circle geometry

Here we will discuss the circle and its properties
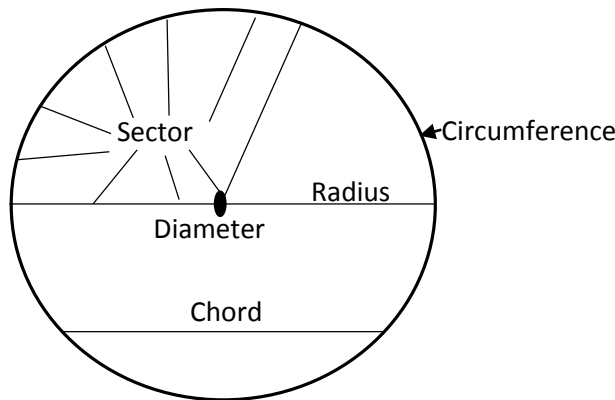
13.1 Important terminologies

1. Circumference - is the distance around the circle. It is the perimeter of the circle.
2. Diameter – is a straight line that divides the circle into two equal parts called semi – circles. A diameter passes through the centre and is from one part of the circumference to the other.

> Circumference = 2 x pi x radius   or   circumference = pi x diameter
> $C = 2\pi r$ $C = \pi d$

3. Radius – is a straight line from the centre of the circle to the circumference. A radius is half of the diameter.
4. Chord – is a straight line that moves from one part of the circumference to the other without passing through the centre of the circle.
5. Sector – is a fraction of a circle that is bound by two radii and part of the circumference.
6. Arc length – is the length of a portion of the circumference in a sector.

> Arc length $= \dfrac{\theta}{360°}\, 2\pi r$ where θ is the angle between the 2 radii in degrees

13.2 Circle properties

(i) The angle opposite the centre in a semi – circle is 90°.
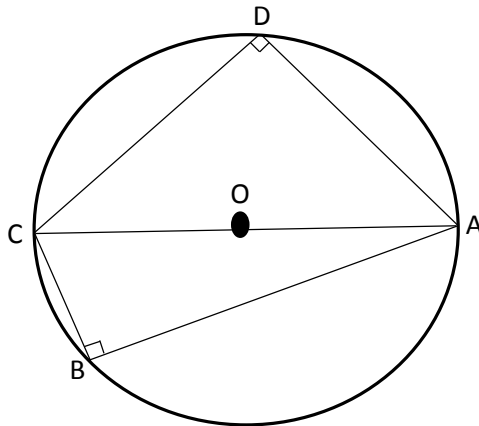
$$\stackrel{\wedge}{ADC} = \stackrel{\wedge}{ABC} = 90°$$

(ii) All angles at the circumference are equal provided that they come from the same chord.
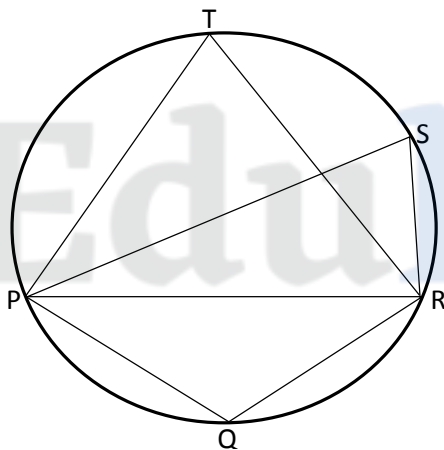
$$\stackrel{\wedge}{PTR} = \stackrel{\wedge}{PSR}$$

(iii)   The angle at the centre is twice the angle at the circumference provided that they come from the same chord.
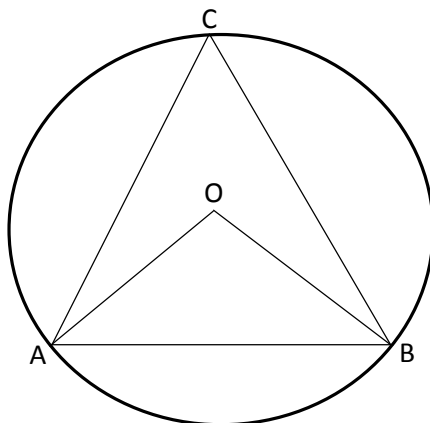
$$\stackrel{\wedge}{AOB} = 2 \stackrel{\wedge}{ACB}$$

(iv) Opposite angles in a cyclic quadrilateral add up to 180°.
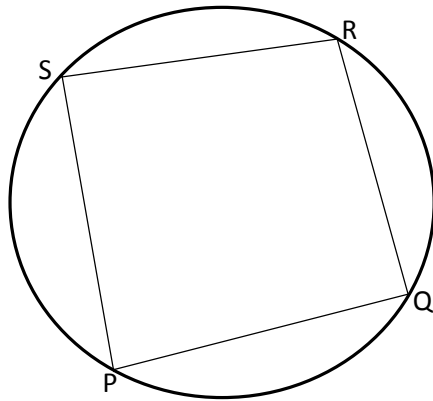
$Q\hat{P}S + Q\hat{R}S = 180°$
$P\hat{S}R + P\hat{Q}R = 180°$

(v) The radius meets the tangent at 90°.

O

Radius
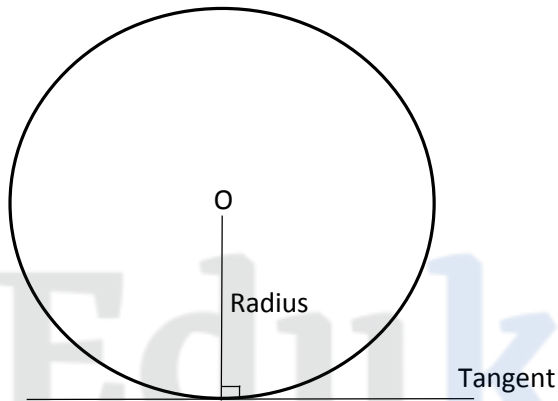
Tangent

(vi) An exterior angle is equal to an interior opposite angle.

A

B

C

$A\hat{B}E = B\hat{D}E$

E

D
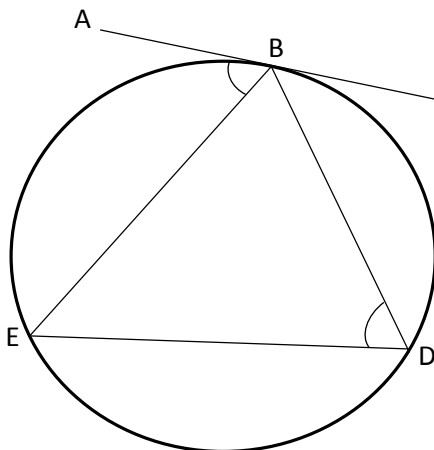
(vii) When two tangents meet, the angle between them and the angle between the 2 radii add up to 180°.
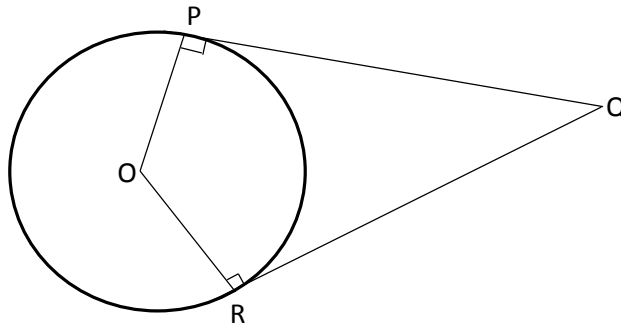
$$\hat{POR} + \hat{PQR} = 180°$$

Examples 1

In the diagram below, ABC is a tangent and O is the centre of the circle. BE is the diameter, $\hat{DEO} = 25°$ and $\hat{FEO} = 22°$.
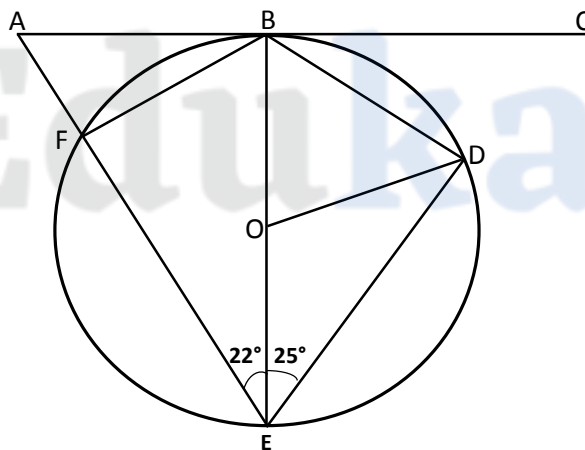
Calculate

(i)	$\hat{BOD}$,	[1]
(ii)	$\hat{CBD}$,	[1]
(iii)	$\hat{DBF}$.	[1]

Solutions		**Reasoning**	
(i) $\hat{BOD} = 2\,\hat{DEO}$ (ii) $\hat{CBD} = \hat{DEO} = 25°$		(i)	Angle at centre = 2 angle at circumference
$\quad = 2 \times 25°$		(ii)	Ext. angle = int. opposite angle
$\quad = \underline{50°}$		(iii)	Sum of opposite angles in cyclic quadrilateral is 180°
(ii) $\hat{DBF} = 180° - \hat{FED}$			
$\quad = 180° - 47°$			
$\quad = \underline{133}$			

Example 2

In the diagram below, P, Q, R and S are points on the circumference of a circle with centre O. AQB is a tangent to the circle at Q.
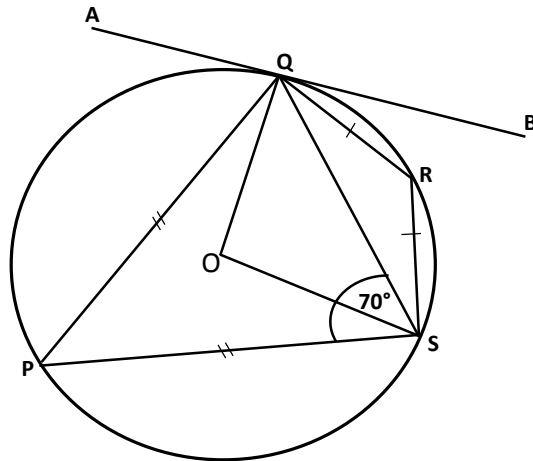
Given that PS = PQ, QR = RS and $P\hat{S}Q = 70°$, calculate

(i)	$Q\hat{P}S$,	[1]
(ii)	$Q\hat{R}S$,	[1]
(iii)	$A\hat{Q}P$,	[1]
(iv)	$S\hat{O}Q$.	[1]

Solutions

(i) $Q\hat{P}S = 180° - P\hat{S}Q - P\hat{Q}S$

      = 180° - 70° - 70°

      = 40°

(ii) $Q\hat{R}S = Q\hat{P}S = \underline{40°}$

(iii) $A\hat{Q}P = P\hat{S}Q = \underline{70°}$

(iv) $S\hat{O}Q = 2\ Q\hat{P}S$

      = 2 x 40°

      = 80°

Reasoning

(i) Triangle PSQ is isosceles. That is angle PSQ = angle PQS

(ii) Angles from the same chord

(iii) Ext. angle = int. opposite angle

(iv) Angle at centre is twice angle at circumference

Example 3

In the diagram below, A, B, C and D are points on the circumference of a circle. Line DC produced meets BF at E. The point O is the centre, DF and BF are tangents to the circle.
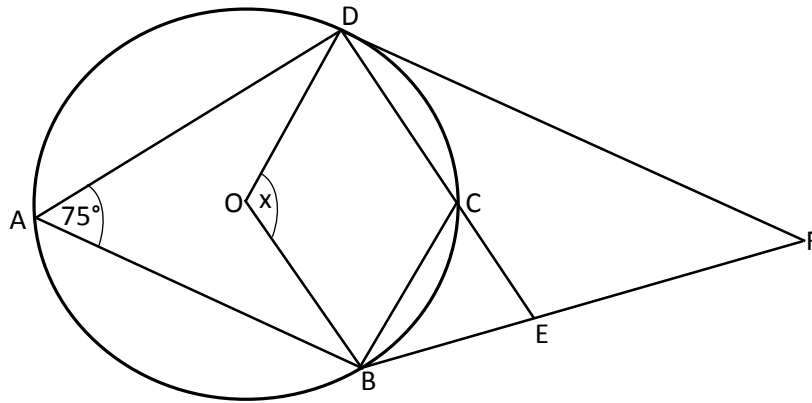
Given that B$\hat{A}$D = 75°, find:

(i)   Angle x.   [1]
(ii)   B$\hat{C}$E.   [1]
(iii)   O$\hat{D}$F.   [1]
(iv)   B$\hat{F}$D.   [2]

<table>
<tr><th colspan="2">Solutions</th><th>Reasoning</th></tr>
</table>

Solutions		Reasoning

(i) x = 2 B$\hat{A}$D
 	
  = 2(75°)
 	
 = <u>150°</u>
	(ii) B$\hat{C}$E = 180° - B$\hat{C}$D
	
	  = 180° - 75°
	
	  = <u>105°</u>
		(i) Angle at centre is twice angle at circumference.
		(ii) B$\hat{A}$D = B$\hat{C}$D = 75° (angles from same chord) and DCE is a straight line (total angle is 180°)
		(iii) Radius meets the tangent at 90°.
		(iv) B$\hat{O}$D + B$\hat{F}$D = 180°

(iii) O$\hat{D}$F = <u>90°</u>	(iv) B$\hat{F}$D = 180° - B$\hat{O}$D
	
	  = 180° - 150°
	
	  = <u>30°</u>	

Example 4

In the diagram below, ABCD is a circle, with centre O. given that $O\hat{A}D = 42°$ and $O\hat{A}B = 36°$,
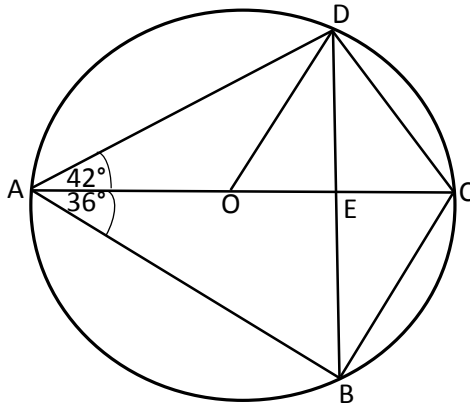
Calculate

(i) $A\hat{O}D$,   [1]
(ii) $A\hat{B}C$,   [1]
(iii) $O\hat{D}B$.   [2]

Solutions

(i) $A\hat{O}D = 180° - O\hat{A}D - O\hat{D}A$ (ii) $A\hat{B}C = \underline{90°}$

      $= 180° - 42° - 42°$

      $= \underline{96°}$

(iii) $O\hat{D}B = A\hat{D}C - O\hat{D}A - C\hat{D}B$

      $= 90° - 42° - 36°$

      $= \underline{12°}$

Reasoning

(i)   Triangle AOD is isosceles. Base angles are equal. Total angles in triangle add up to 180°.

(ii)   Angle opposite centre in semi – circle = 90°.

(iii) $C\hat{D}B = B\hat{A}C = 36°$ (angles from same chord). $A\hat{D}C = 90°$

Task

In the diagram below, A, B, C and D are points on the circumference of a circle. TCM and MDE are tangents to the circle at C and D respectively.
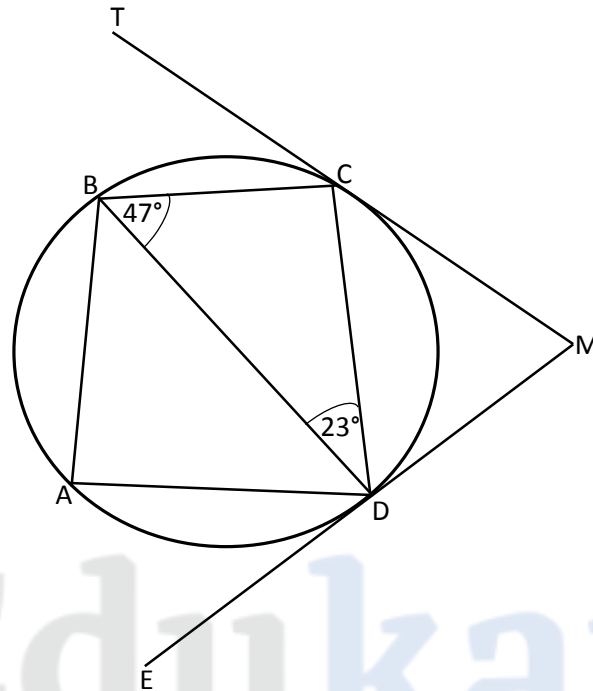
Given that AB = AD, $C\hat{B}D = 47°$ and $B\hat{D}C = 23°$, calculate

(i)	$B\hat{C}T$,	[1]
(ii)	$B\hat{A}D$,	[1]
(iii)	$A\hat{B}D$,	[1]
(iv)	$D\hat{M}C$.	[2]

14. Trigonometry

A triangle is a shape with three sides and three interior (inside) angles. All the interior angles adds up to 180°. This chapter will deal with triangles and their properties.

14.1 Types of triangles

1. **Right angled triangle**. This is a type of triangle in which one of the interior angles in 90°.

(a) Area

$$Area = \frac{1}{2} \ base \ x \ height$$

$$A = \frac{1}{2} \ bh$$

Height

Base
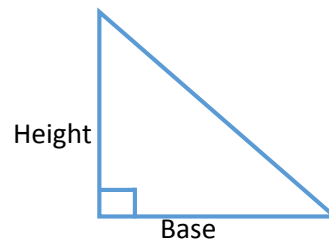
(b) The Pythagoras theorem

Hypotenuse² = adjacent² + opposite²

$c^2 = a^2 + b^2$

(c) Trigonometric ratios

$$sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Opposite

Hypotenuse

Adjacent

θ
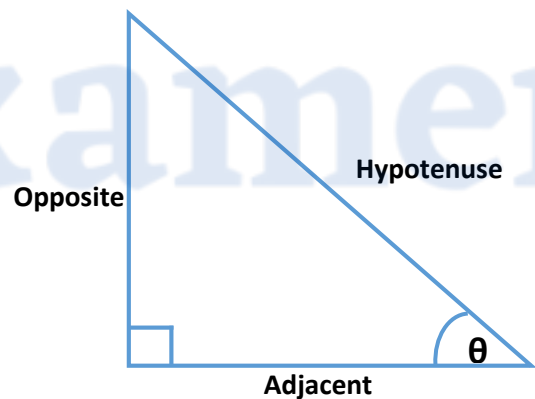
To remember these trigonometric ratios use the mnemonic **SOHCAHTOA**

S for sinθ, O for opposite, H for hypotenuse, C for cosθ, and T for tanθ

NOTE. The Pythagoras theorem and trigonometric ratios are only applicable to a right angled triangles.

Relationship between the sine and cosine of exterior and interior angles

cos interior angle = - cos exterior angle

sin interior angle = sin exterior angle

2. **Isosceles triangle**. In this type of triangle two of the sides have equal lengths and two interior base angles are also equal.
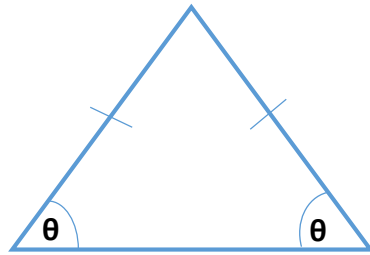
3. **Equilateral triangle**. In this triangle the lengths of all the sides are equal and all the interior angles are equal. Each interior angle is 60°.
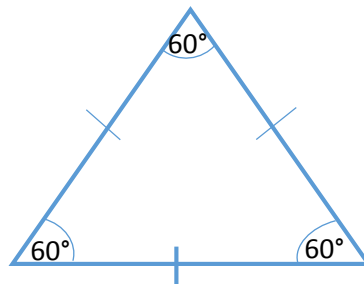
4. **Scalene triangle**. This is a type of triangle in which all the sides are of different lengths and the interior angles are also different.
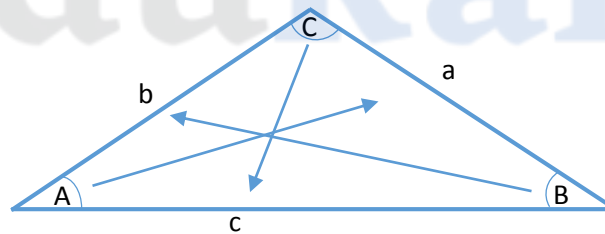
(a) Area

$$Area = \frac{1}{2}\, ab\, sinC$$

(b) Cosine rule

When you have been given the lengths of two sides and an angle between them, use the cosine rule to find the length of the other side. This rule can also be used to calculate the angles inside if and only if the lengths of all three sides have been given.

$$c^2 = a^2 + b^2 - 2abcosC$$

Where c is the length to be calculated, **a** and **b** are the given lengths and **C** is the angle between **a** and **b**.

(c) Sine rule

You can use the sine rule to find the angle between two sides given the lengths of both sides and an angle opposite one of the given sides. Also, it can be used to find the length of one of the sides of an isosceles triangle given the value of the angle opposite that side and the length of one of the sides with its opposite angle given.

$$\frac{a}{Sin\ A} = \frac{b}{Sin\ B} = \frac{c}{Sin\ C}$$

(d) Shortest distance (h)

Shortest distance refers to the height (h) of an isosceles triangle. It is given by:

$$h = \frac{2A}{b}$$

where A is the area of the triangle and b is the base length

14.2 Bearings

Bearings involves the description of direction or geographical position in terms of angles in degrees. The commonly used method of description is the three figure bearing, which uses three digits.

0° is 000°

1° is 001°

10° is 010°

100° is 100°

150° is 150°

14.21 How to find bearings

Consider two towns A and B. we can find the bearing of A from B or B from A by firstly joining the two places with a straight line. **To find the bearing of A from B.**

- ✓ Draw the true north at B.
- ✓ Move in clockwise direction from the north until you touch the line joining the two places.
- ✓ Measure the angle formed using a protractor if the diagram is drawn to scale otherwise calculate it.

To find the bearing of B from A.
- ✓ Draw the true north at A.
- ✓ Move in a clockwise direction from the north until you touch the line joining the two places.
- ✓ Calculate the angle formed or measure it if the diagram is drawn to scale.
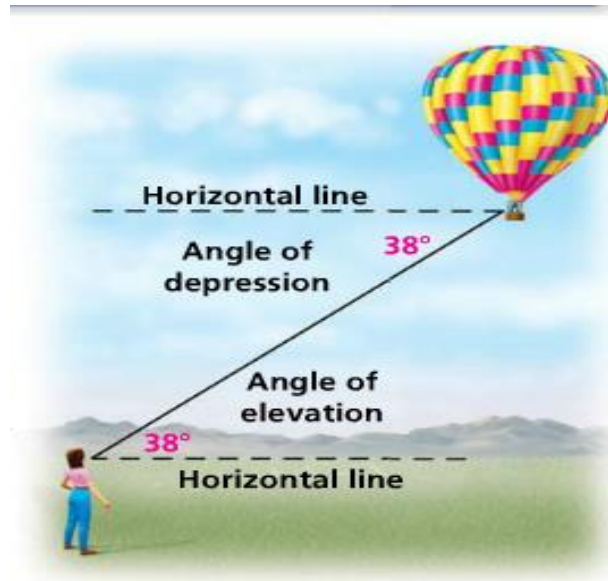
Angle of elevation is the angle between the horizontal and the line joining the observer and the object when the observer views an object that is above.

Angle of depression is the angle between the horizontal and the line joining the observer and the object when the observer views an object that is below.

Example 1

In the diagram below, triangles ABC and ACD are right angled. $A\hat{B}C = C\hat{A}D = 90°$, AC = 20.2 cm, BC = 9.5cm and AD = 12 cm.
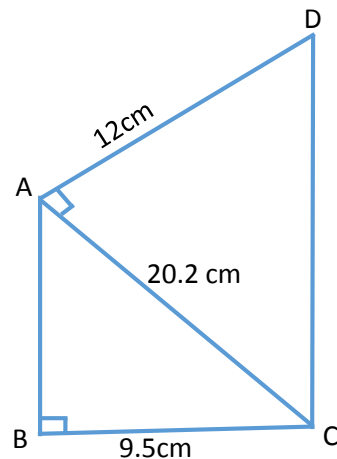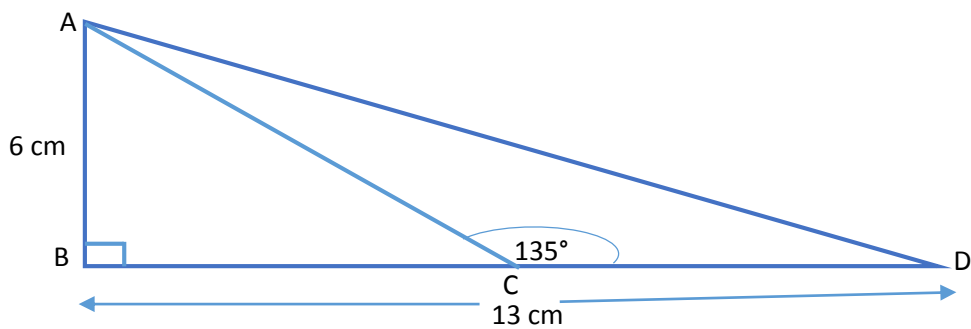

Calculate

(a) AB,   [2]

(b) B$\hat{A}$C,   [2]

(c) The area of quadrilateral ABCD,   [3]

(d) The Perimeter of quadrilateral ABCD. [3]

Solution	Reasoning
(a) $AB = \sqrt{(AC)^2 - (BC)^2}$ $= \sqrt{(20.2cm)^2 - (9.5cm)^2}$ $= \sqrt{317.79cm^2}$ $= \underline{17.8\ cm}$	(a) Since triangle ABC is right angled, use Pythagoras theorem. AC is hypotenuse.
(b) $Sin\ B\hat{A}C = \frac{BC}{AC}$ $Sin\ B\hat{A}C = \frac{9.5\ cm}{20.2\ cm}$ $\sin^{-1} B\hat{A}C = 0.470297029^{-1}$ $B\hat{A}C = \underline{28°}$	(b) Use any trig. Ratio. AB is adjacent, BC is opposite, AC is hypotenuse and θ is angle BAC.
(c) Area of ABCD = Area of ABC + Area of ACD $= \frac{1}{2}(BC)(AB) + \frac{1}{2}(AC)(AD)$ $= \frac{1}{2}(9.5cm)(17.8cm) + \frac{1}{2}(20.2cm)(12.0cm)$ $= 84.55\ cm^2 + 121.2cm^2$ $= \underline{205.8\ cm^2}$	(c) To find total area, calculate the areas of triangles ABC and ACD and the two areas.
(d) $CD = \sqrt{(AD)^2 + (AC)^2}$ $= \sqrt{(12cm)^2 + (20.2cm)^2}$ $= \sqrt{552.04cm^2}$ $= 23.5\ cm$ Therefore, $P = AB + BC + CD + AD$ $= 17.8cm + 9.5cm + 23.5cm + 12.0cm$ $= \underline{62.8\ cm}$	(d) Find DC first using the Pythagoras theorem. DC is hypotenuse. Finally, add all the lengths making the boundary of ABCD

Example 2

In the diagram below, BCD is a straight line, AB = 6 cm, BD = 13 cm, $A\hat{C}D = 135°$ and $A\hat{B}C = 90°$.

(a) Calculate the length of CD. [1]
(b) Calculate the area of triangle ACD. [2]

Solution

(a) AB = BC = 6 cm, since BAC = BAC = 45°, triangle ABC is isosceles.
Therefore, CD = BD − BC
$$= 13cm − 6cm$$
$$= \underline{7cm}$$

(b) Area of triangle ACD = area of triangle ABD − area of ABC
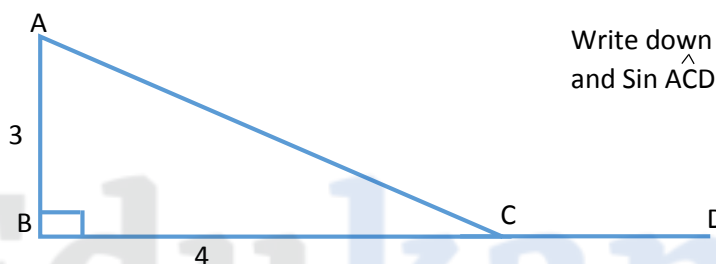$$= \frac{1}{2}(BD)(AB) - \frac{1}{2}(BC)(AB)$$
$$= \frac{1}{2}(13cm)(6cm) - \frac{1}{2}(6cm)(6cm)$$
$$= 39 \text{ cm}^2 - 18\text{cm}^2$$
$$= \underline{21 \text{ cm}^2}$$

Example 3

In the diagram below ABC is a right angled triangle and BCD is a straight line. AB is 3 m and BC is 4 m.

Write down the values of Cos $A\hat{C}D$ and Sin $A\hat{C}D$.   [3]

Solutions

$$AC = \sqrt{(AB)^2 + (BC)^2}$$
$$= \sqrt{(3m)^2 + (4m)^2}$$
$$= \sqrt{25m^2}$$
$$= 5 \text{ m}$$
$$Cos\ A\hat{C}B = \frac{BC}{AC} = \frac{4}{5}$$

$$Cos\ A\hat{C}D = -Cos\ A\hat{C}B = -\frac{4}{5}$$

Also,

$$Sin\ A\hat{C}B = \frac{AB}{AC} = \frac{3}{5}$$

$$Sin\ A\hat{C}D = Sin\ A\hat{C}B = \mathbf{\frac{3}{5}}$$

Reasoning

✓ ACD is exterior angle while ACB is interior angle.
✓ Cos ext. angle = -Cos int. angle
✓ Sin ext. angle = Sin int. angle
✓ Find ratio of int. angle first & use it to find ratio of ext. angle

Example 4

To increase the number of people accessing anti – retroviral drugs (ARVs), an organisation established 3 voluntary and counselling centres in one District. The centres were called A, B and C as shown in the diagram below.
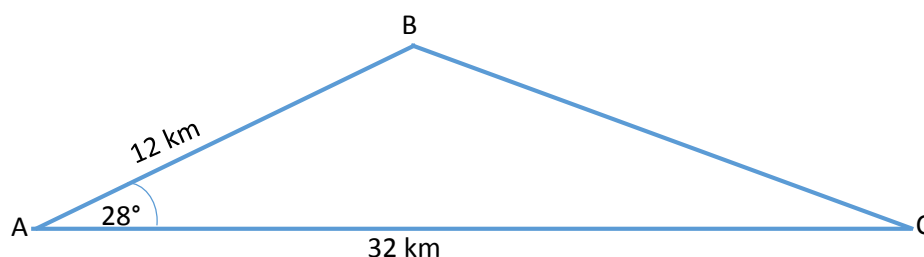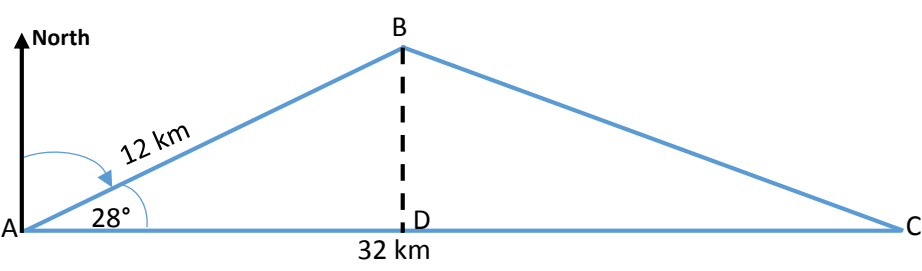
Given that C is due east of A, $B\hat{A}C$ = 28°, AB = 12 km and AC = 32 km, calculate

    (i)   The bearing of B from A   [1]

    (ii)   The distance between B and C   [4]

The organisation decided to put up another centre D due east of A and closest to B.

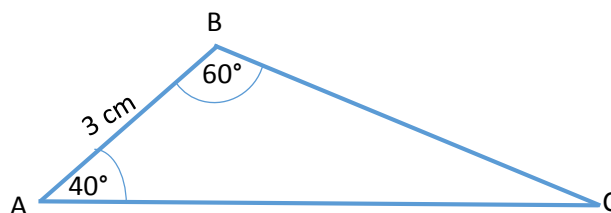   (iii)   Calculate distance BD.   [2]

Solution

(i)   Bearing of B from A = 090° - 28° = <u>062°</u>

(ii) $(BC)^2 = (AB)^2 + (AC)^2 - 2(AB)(AC)Cos\,B\hat{A}C$
            $(BC)^2 = (12km)^2 + (32km)^2 - 2(12km)(32km)Cos\,28°$
            $BC = \sqrt{489.8962km^2}$
            $BC = \underline{22km}$

(iii)   Area of triangle ABC $= \frac{1}{2}(AB)(AC)Sin\,B\hat{A}C$
                            $= \frac{1}{2}(12km)(32km)Sin28°$

                            $= 90\ km^2$

        Therefore, BD $= \frac{2A}{AC} = \frac{2\ x\ 90km^2}{32km} = \underline{6km}$

Reasoning

(i)   Angle between North Pole and AC is 90°.

(ii)   Cosine rule

(iii)   BD is shortest distance. Find area of triangle and use it to calculate height.

Example 5

In the diagram below ABC is a triangle in which AB = 3 cm, $A\hat{B}C$ = 60° and $B\hat{A}C$ = 40°.

Calculate the length BC. [3]

Solution

$$\frac{BC}{Sin\ BAC} = \frac{AB}{Sin\ ACB}$$

$$\frac{BC}{Sin\ 40°} = \frac{3\ cm}{Sin\ 80°}$$

BC = $\frac{3\ cm\ x\ Sin\ 40°}{Sin\ 80°}$
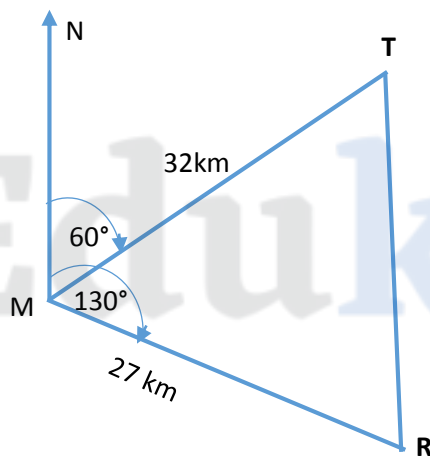
BC = **2 cm**

Reasoning

Use sine rule since 1 side and 2 angles are given.

Use ratio of side to the sine of the angle opposite

Task

M, T and R are Food Reserve Agency Maize buying points in a given district. T is 32 km from M on a bearing of 060° and R is 27 km from M on a bearing of 130° as shown in the diagram.

Calculate

    (i)   Angle RMT, [1]
    (ii)   The area of triangle MTR, [2]
    (iii)   TR to the nearest kilometre, [5]
    (iv)   MS, given that there is a shopping centre S along RT such that MS is the shortest
              distance from M. [2]

15. Mensuration

Mensuration basically deals with the measurements of area, lengths (e.g. perimeter) and volume of objects of different shapes. The objects usually dealt with have regular shapes. The shapes that will be discussed here are the square, rectangle, circle, semi – circle, cylinder, cone and pyramid.

15.1 Square.
A square has 4 sides of equal lengths.
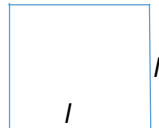
Area = length x length

$$A = l^2$$

Perimeter = $l + l + l + l$

P = 4l

> **A square of length 4m will have an area of 16m² and a perimeter of 16m.**

15.2 Rectangle
A rectangle is a four sided figure that is longer than it is wide. Opposite sides have equal length.

breadth

length

Area = length x breadth

$$A = lb$$

> **A rectangle of length 10cm and breadth of 5 cm will have an area of 50cm² and a perimeter of 30cm**

Perimeter = 2l + 2b

15.3 Circle
A circle round/ringed shape.
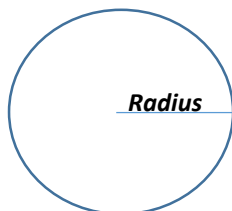
Radius

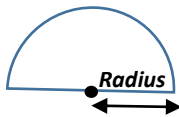A circle of radius 2m will have an area of 12.6m² and a perimeter/circumference of 12.6m

Area = πr²

Perimeter

Remember, the perimeter of the circle is its circumference.

| $C = 2\pi r$ | or | $C = \pi d$ |

15.5 Semi-circle

This is a half circle.

Radius

| **Area** $= \dfrac{1}{2} \pi r^2$ |

15.6 Cylinder

A cylinder is a uniform tubular shape with circular ends.

A.

B.
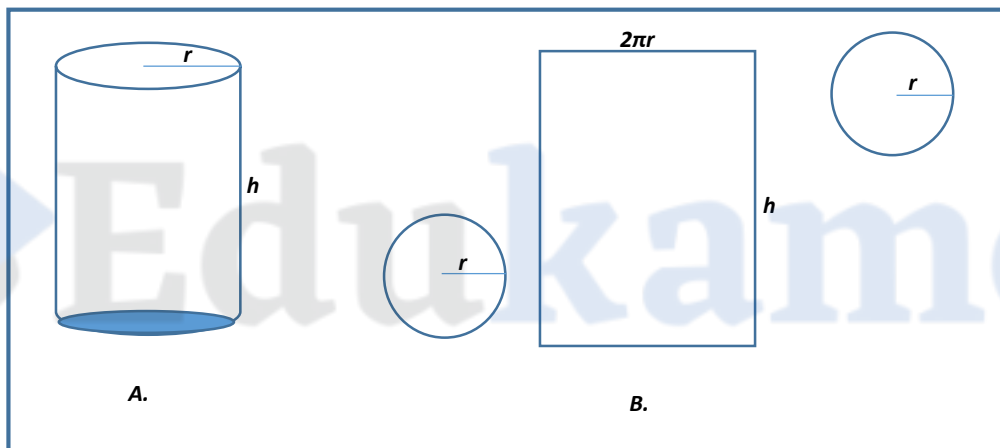
When the cylinder is split as in figure B above, it is composed of two circles that are of the same size and a rectangle. The length of this rectangle is the height of the cylinder and the breadth is the circumference of the circle (i.e. $2\pi r$).

Types of cylinder

    a.   Closed cylinder – is one which has both circular ends closed. That is both lids are present. For example, a canned drink before it is opened.

| Area of closed cylinder = $2\pi rh + 2\pi r^2$ | **where h is the height of the cylinder**

    b.   Open cylinder – is one which has both lids removed such that one can even see through.

| Curved surface area (CSA) = $2\pi rh$ |

Volume of a cylinder

| $V = \pi r^2 h$ | that is, volume = base area x height

15.7 Cone and pyramid
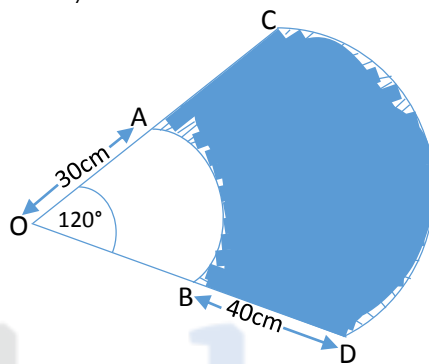
The volume of a cone or pyramid is given by

$$V = \frac{1}{3} \text{ base area x height}$$

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3$$

Example 1

The shaded part of the diagram below is that of a path of a car windscreen wiper. The wiper rotates through 120° about O. (take π to be $\frac{22}{7}$).

Given that OA = OB = 30cm, AC = BD = 40cm, calculate

(i)   The perimeter of the sector OCD correct to 1 decimal place,   [3]
(ii)   The area of the shaded region correct to 1 decimal place.   [3]

Solution

(i)   First find the arc length CD. CD is is part of the circumference subtended by an angle of 120° and radius of 70cm.

$$CD = \frac{\theta}{360°} 2\pi r$$

$$CD = \frac{120°}{360°} \text{ x } 2 \text{ x } \frac{22}{7} \text{ x } 70cm$$

CD = 146.7cm

Therefore, Perimeter = OD + OC + CD

                        = 70cm + 70cm + 146.7cm

                         = 286.7 cm

(ii)   The radius(R) of sector OCD = 70cm and the radius (r) of sector OAB = 30cm

Area of shaded region = area of sector OCD − area of sector OAB

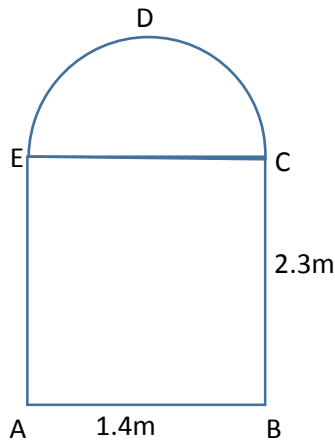$$= \frac{\theta}{360}\pi R^2 \ - \ \frac{\theta}{360}\pi r^2$$

$$= \frac{120°}{360°} \text{ x } \frac{22}{7} \text{ x } (70cm)^2 \ - \ \frac{120°}{360°} \text{ x } \frac{22}{7} \text{ x } (30cm)^2$$

$$= 5133.3 \text{ cm}^2 - 942.9cm^2$$

$$= 4190.4cm^2$$

Example 2

The diagram below represents a metallic window frame of a sports hall. It consists of a rectangular frame ABCE and a semi – circular arc CDE. The two parts are welded together at point C and E.

Given that AB = 1.4m and BC =2.3m, (take π to be $\frac{22}{7}$).

(a) Calculate:
(i)   The length of the metal CDE.   [2]
(ii)   The total length of the metal required to make the frame ABCDE.   [2]
(b) Glass panes would be required to cover this window frame. Find the area of the total sheet of glass panes required.   [3]
(c) Given that glass costs K42.5 per square metre, calculate how much was spent on glass panes for the window.   [2]

Solution

(a) (i) CDE = πr   **note that CDE is half the circumference. EC is the diameter**

$= \frac{22}{7}$ x 0.7m   **= AB = 1.4 m. Radius = 1.4m ÷ 2 = 0.7 m**.

= 2.2 m

(ii) Total Length (perimeter) = AB + BC + CDE + EA

= 1.4m + 2.3m + 2.2m + 2.3m

= 8.2m

(b) Total area = area of semi – circle CDE + area of rectangle ABCE

$= \frac{πr^2}{2}$ + (AB x BC)

$= \frac{\frac{22}{7}x\,(0.7m)^2}{2}$ + (1.4m x 2.3m)

= 1.54m² + 3.22m²

= 4.76m²

(c) Total cost = K42.5 x 4.76

= K202.3

Example 3

The diagram below represents a glass of water of uniform cross section and thickness.
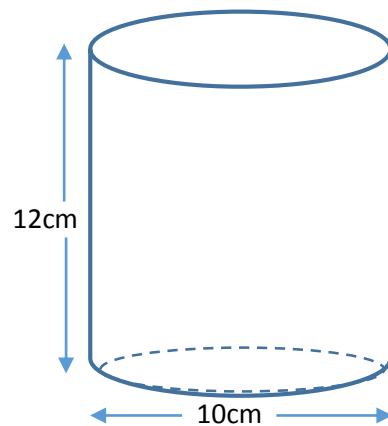
Given that its height is 12cm, diameter is 10cm and taking π = 3.142,

(i) Calculate the outer surface area of the glass correct to 2 decimal places. [3]
(ii) Given that the thickness of the glass of water is 2mm, calculate the volume of water that the glass can hold when completely full, giving your answer correct to 2 decimal places. [4]

Solution

(i) Area = 2πrh + πr²
       = (2 x 3.142 x 5cm x 12cm) + (3.142 x 5cm x 5cm)
       = 377.04cm² + 78.55cm²
       = 455.59cm²

(ii) Volume = πr²h
         = 3.142 x (5cm)² x 12cm
         = 942.60cm³

Task

A cylindrical geyser of radius 21 cm and length 50 cm is placed with its curved surface on a horizontal ground. It is filled partially with water and the segment ABY in the diagram shows the cross section of water in the geyser. O is the centre of the circular end of the geyser, Y is vertically below O and angle AOB = 120°. (take $\pi = \frac{22}{7}$).
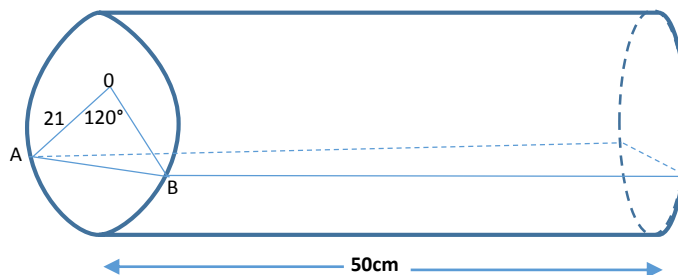

Calculate

(i)   The curved surface area of the geyser,   [2]

(ii)   The volume of water in the geyser correct to the nearest whole number.   [4]

16.   Earth Geometry

Here we will look at the geometry of the earth. This will include describing locations of places in terms of longitude and latitudes, time calculation and distance calculation.

The earth is a sphere. It is divided into north and south vertically and west and east horizontally. The radius (R) of the earth is 6370 km or 3437 nautical miles (nm).
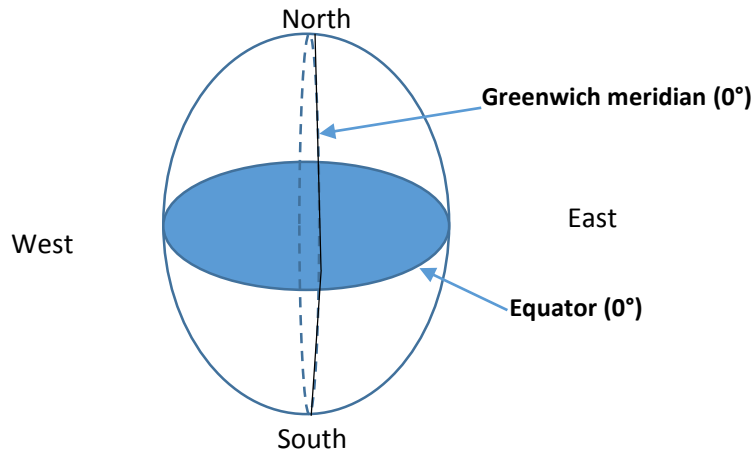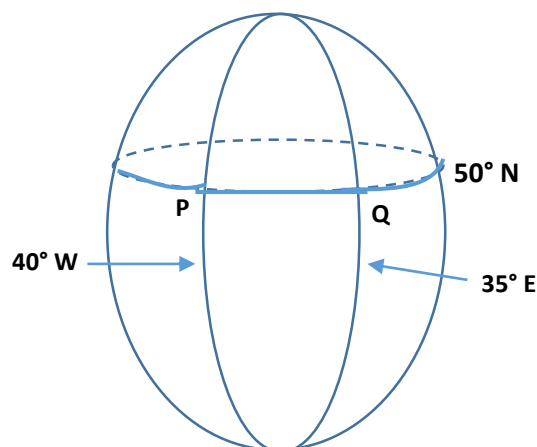

16.1 Important terms

Latitude. These are lines that run from west to east. The equator is the longest latitude and it separates north from south.

Longitude. These are lines that run from north to south. The Greenwich meridian is the longest longitude and it separates west from east.

Locating places on the surface of the earth

To locate a place state the latitude and longitude it is lying on. For example, in the diagram above, P is (50°N, 40°W) and Q is (50°N, 35°E).

16.2 Calculation of time

Time is measured either east or west. There is gain in time when moving from west to east and vice versa. That is going east you gain time, going west you lose time. Every 15° moved west or east represent 1 hour. That is if you move 15° east you gain 1 hour, 30° 2hours and 1f you move 15° west you lose 1 hour, and 2 hours is lost for every 30° moved.

Example

At place X (30°N, 30°E) People are watching a live football match at 18: 00 hours. At place Y (30°N, 60°W) the same match is been broadcasted live. What time is this match been broadcasted at Y?

<div align="center">

Solution

</div>

First find the difference in longitude. If the places are on the same side of the compass e.g. both on the west or both on the east subtract the smallest longitude from the largest longitude. If they are on opposite sides (e.g. west and east) add the longitudes. So in the example above, the difference in longitude is

$$30° + 60° = 90°$$
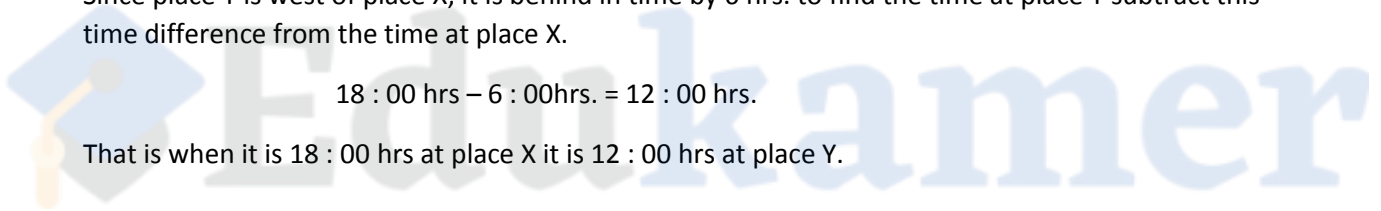
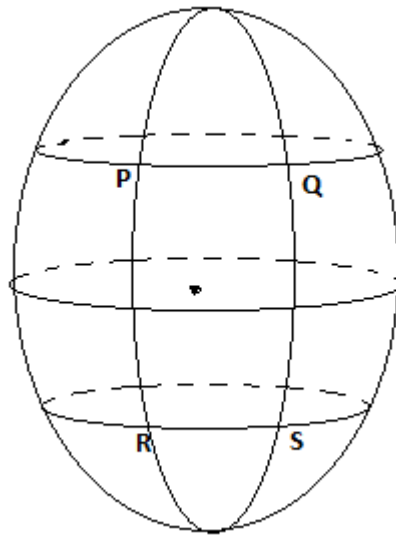Divide this difference by 15° to find the time difference between the two places.

$$\frac{90°}{15°} = 6 \quad \text{hours}$$

Since place Y is west of place X, it is behind in time by 6 hrs. to find the time at place Y subtract this time difference from the time at place X.

$$18 : 00 \text{ hrs} - 6 : 00 \text{hrs.} = 12 : 00 \text{ hrs.}$$

That is when it is 18 : 00 hrs at place X it is 12 : 00 hrs at place Y.

16.3 Calculating distance along the parallel of longitude and parallel of latitude

The distance along the parallel of longitude (e.g. QS) is given by

$$distance(QS) = \frac{\theta}{360}\, 2\pi R$$

Where R is the radius of the earth and θ is the difference in latitude.

The distance along the parallel of latitude (e.g. PQ) is given by

$$distance(PQ) = \frac{\theta}{360}\, 2\pi r \cos L$$

Where L is the latitude on which the places lie and θ is the difference in longitude.
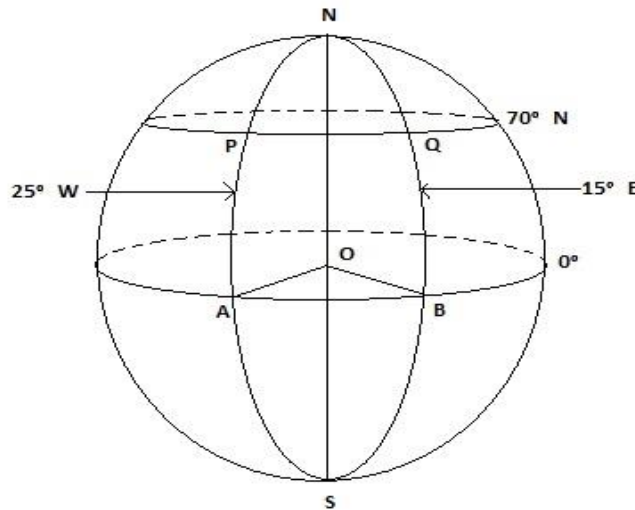
16.4 Calculation of the speed

$$speed = \frac{distance}{time}$$

If distance is in kilometres, the units for speed is kilometre per hour (km/hr). if distance is in nautical miles, speed is given in knots.

Example

P and q are points on the surface of the earth situated on the same parallel of latitude 70°N as shown below. The longitudes of P and Q are 25° and 15°E respectively. A and B are two points on the equator such that A is due south of P and B is due south Q. (π = 3.142, R = 3437nm).

(i)   State the position of the point Q.
(ii)   find the distance between A and B
(iii)   Given that the time at B is 09: 20 hours, what is the time at A.?
(iv)   Calculate the circumference of the small circle at 70°N.
(v)   Find the distance along the parallel of latitude between P and Q, correct to 2 decimal places.
(vi)   Find the distance along the parallel of latitude between P and A, correct to 2 decimal places.

Solution
(i)   Q (70°N, 15°E)
(ii)   This is distance along parallel of latitude, the difference in longitude between A and B is 15°+25° = 40°. Thus θ = 40° and L= 0°. substituting these in the formula

$$D_{(A \& B)} = \frac{\theta}{360} 2\pi R \cos L$$

$$= \frac{40°}{360°} \times 2 \times 3.142 \times 3437nm \times \cos 0°$$

= 2399.79nm

(iii)   The difference in longitude between A and B is 40°. To find time difference divide 40° by 15° because 1 hour is equivalent to 15°. Therefore time difference $= \frac{40°}{15°} = \frac{8}{3} = 2:40$ hours. Since B is east it is ahead in time by 2 : 40 hours. To find the time at A subtract 2 : 40 hours from the time at B.

Thus time at A = 09: 20 hours – 2 : 40 hours = **06 : 40 hours**

(iv)   C = 2πRcosL   here L = 70°
        = 2 x 3.142 x 3437nm x cos70°
        =7386.99nm

(v)   This is distance along parallel of latitude, the difference in longitude between P and Q is 15°+25° = 40°. Thus θ = 40° and L= 70°. Substituting these in the formula

$$D_{(A \& B)} = \frac{\theta}{360} 2\pi R \cos L$$

$$= \frac{70°}{360°} \times 2 \times 3.142 \times 3437nm \times \cos 70°$$

$$= \textbf{1436.36nm}$$

(vi)   This is distance along parallel of longitude, the difference in latitude between P and A is 70°+0° = 70°. Thus θ = 70°. substituting these in the formula

$$D_{(A \& B)} = \frac{\theta}{360} 2\pi R \cos L$$

$$= \frac{70°}{360°} \times 2 \times 3.142 \times 3437nm$$

$$= \textbf{4199.63 nm}$$

Task 1

(a) Compute the difference in latitude of the two towns Lundazi (12.9°S, 33°E) and Khartoum (15.4°N, 33°E).

(b) given that the earth is a sphere of radius 6370km and π= 3.142, calculate

    (i)   The length of the equator

    (ii)   The length of the meridian from North Pole to the South Pole through Greenwich.

Task 2

The diagram below shows a model of the earth. The points C and D are on the same longitude. The latitudes of C and D are 45°N and 55°S respectively. (Take π = 3.142 and R = 3437nm).
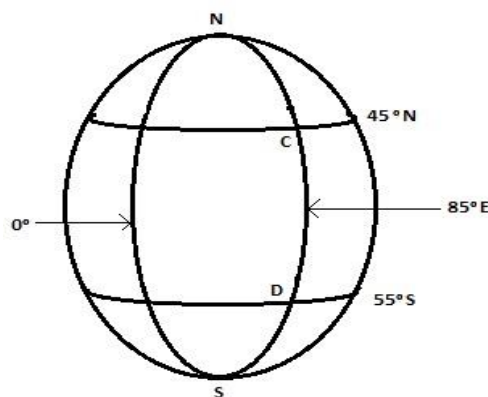
(i)   Write the position of the point C.

(ii)   Calculate the difference in latitude between C and D.

(iii)   Find the distance CD in nautical miles.

(iv)   Calculate the circumference of the latitude 45°N in nautical miles.

17.   Construction and loci

The key instruments used in constructing geometric shapes are pencil, ruler, compass and protractor.

17.1 Constructing a shape e.g. triangle

Construct ΔABC in which AB = 6cm, AC = 3cm and BC = 4cm

First draw a small sketch of how the triangle will look like, choosing AB as the base.

   a.   Draw the line AB which should be 6cm using a ruler and label the end points as A and B.
   b.   Using a pencil clamped in compass, measure a radius of 4cm, take B as centre and draw an arc.
   c.   Measure another radius of 3cm, take A and draw an arc.
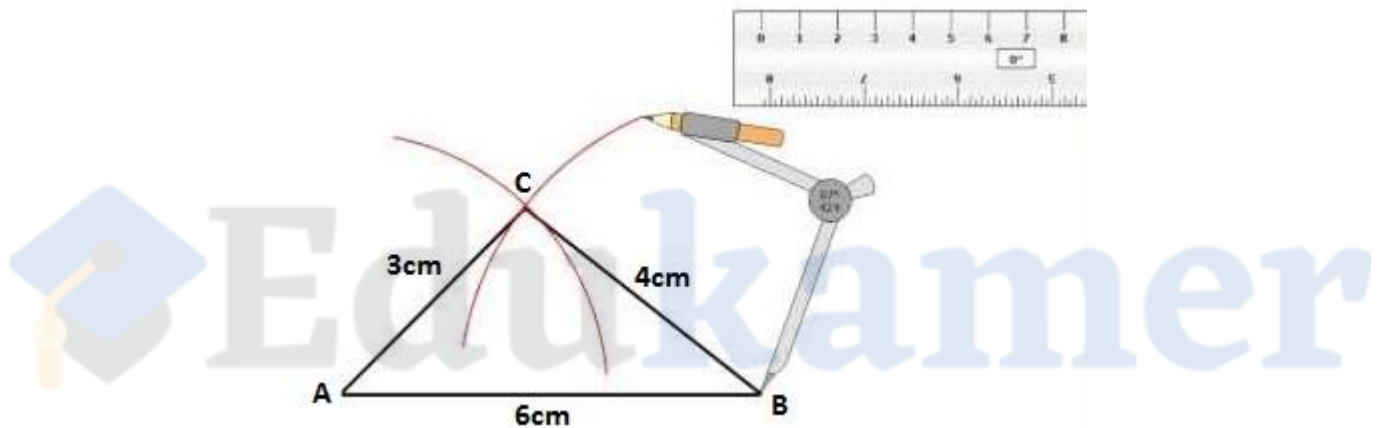   d.   The point where the two arcs intersect is where C is. Join C to A and C to B.

Figure 17.1

17.2 Locus

Locus refers to the set of all points that satisfy a given condition or set of conditions. A point is a locus if and only if it satisfies the given conditions of the locus. A locus is a line or a curve, formed by points that all satisfy a certain condition. A locus can be drawn such that: Its distance from a fixed point is constant, It is equidistant from two given points, Its distance from a given line is constant and It is equidistant from two lines.

   (a) **Locus of points equidistant from to A and B**.
       The locus of points equidistant from two points is the perpendicular bisector of the given line, in this case line AB.
       Measure the radius more than half the radius of AB. Centre at A and draw two arcs one above line AB and the other below line AB.
       Without changing the radius, centre at B and draw an arc above line AB bisecting the first arc. Draw another arc below line AB bisecting the second arc.
       Label the point C where the two arcs above intersect each other and label as D the point where the lower arcs intersect.
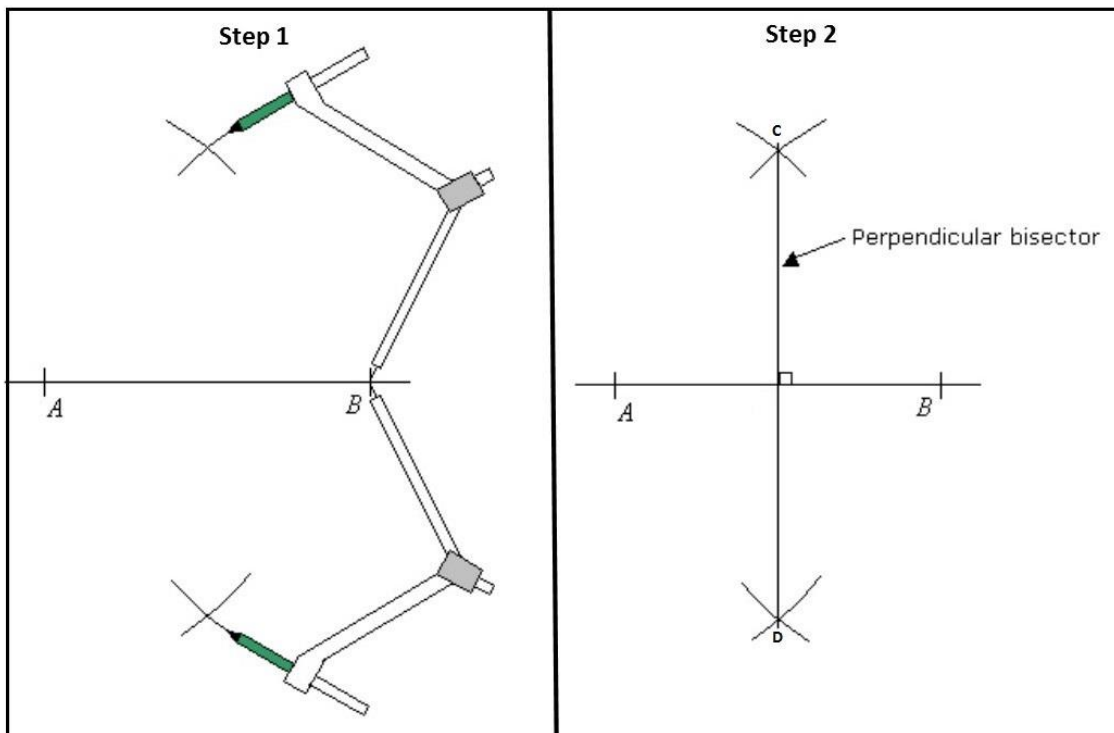   Join C and D. CD is the required perpendicular bisector

Figure 17.21

(b) Locus of points which are 4cm from a fixed point O.
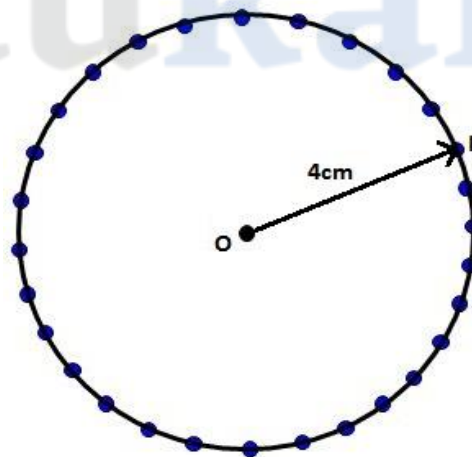   This is a circle with centre O and radius 4cm.

Figure 21.22

The locus of points can lie inside a region. The region of points can be drawn such that:
The points are greater than or less than a given distance from a fixed point.
The points are closer to one given point than to another point

(c) locus of points equidistant from two given line
    This is an angle bisector. e.g. Bisect angle ABC.
    Take any radius, centre at B and draw an arc to cut the line BA and label that point p.
    without changing the centre and radius draw another arc to cut the line BC and label that
    point q.

Draw an arc centre p, radius Bp and draw another arc centre q and radius Bp. Label the point where these two arcs intersect at S.
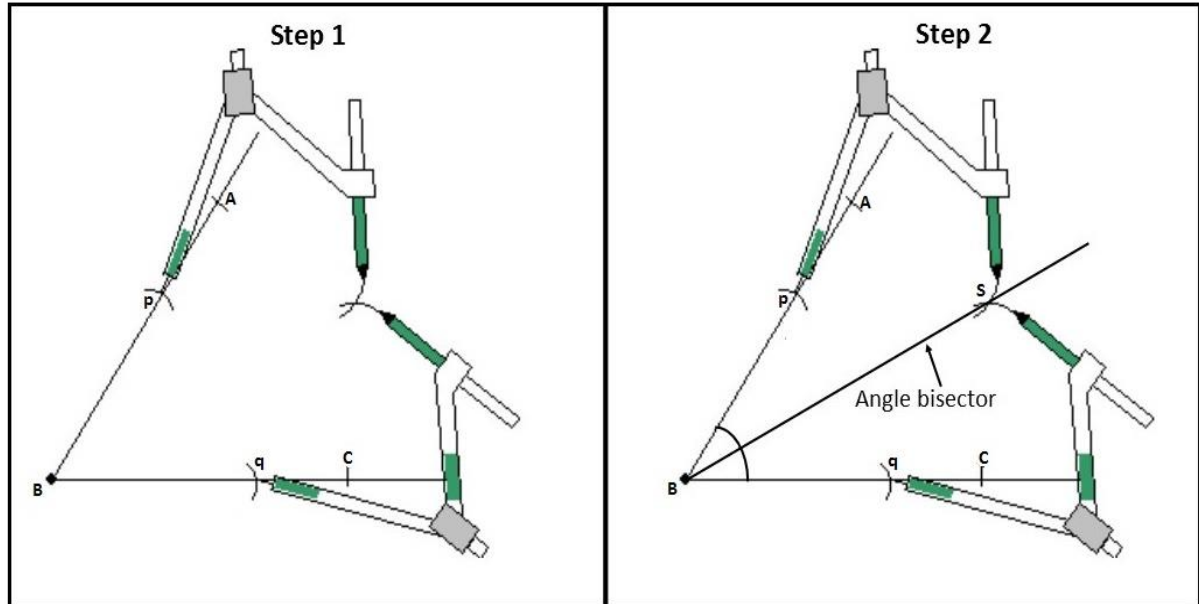Join B to S to get the angle bisector

Figure 17. 23

17.3 Scale drawing and maps

The area or length of very large objects, land and roads can be represented on paper using scale drawing and maps. A certain quantity on the map represent an actual quantity on the ground. to understand the subject fully, the following length and area relationship should be remembered.

Length

- ✓ 1cm = 10mm
- ✓ 1m = 100cm
- ✓ 1km = 1000m
- ✓ 1km = 1000 000cm

Area

- ✓ $1cm^2 = 100mm^2$
- ✓ $1m^2 = 10\,000cm^2$
- ✓ $1km^2 = 1.0 \times 10^6 m^2$
- ✓ $1km^2 = 1.0 \times 10^{10} cm^2$

Example

A map of a game park is drawn to a scale of 1 : 50 000.

(a) Two game camps A and B are 7 cm apart on the map. Find the actual distance between the two parks, in kilometres.
(b) The actual area of the game park is 25 square kilometres (km^2). Calculate the area of the park on the map, in square centimetre.

A scale of 1 : 50 000 means that 1cm on the map represent 50 000 cm on the actual ground. Also, 1 cm^2 on the map represent 2.5 x 10^9 cm^2

(a) let the actual distance between A and B be y

1cm $\longrightarrow$ 50 000cm

7cm $\longrightarrow$ y   **after cross multiplying**

y = 350 000cm = 0.35km   **divide by 1000000 to convert to kilometre**

(b) let x represent the area on the map

1cm^2 $\longrightarrow$ 2.5 x 10^9 cm^2

x $\longrightarrow$ 2.5 x 10^{11}cm^2 (25km^2 =2.5 x 10^{11}cm^2)

after cross multiplying and dividing

x = 2.5 x 10^{2}cm^2 = **2.5 x 10^{-8} km^2**

Example

Answer the whole of this question on a sheet of plain paper.

(a) (i) Construct a quadrilateral PQRS in which PQ = 8 cm, QR = 6 cm,   angle PQR = 110°, RS = 4cm and PS = 9cm.
(ii) Measure and write down the size of angle PSR.
(b) On the same diagram and within quadrilateral PQRS, construct the locus of points
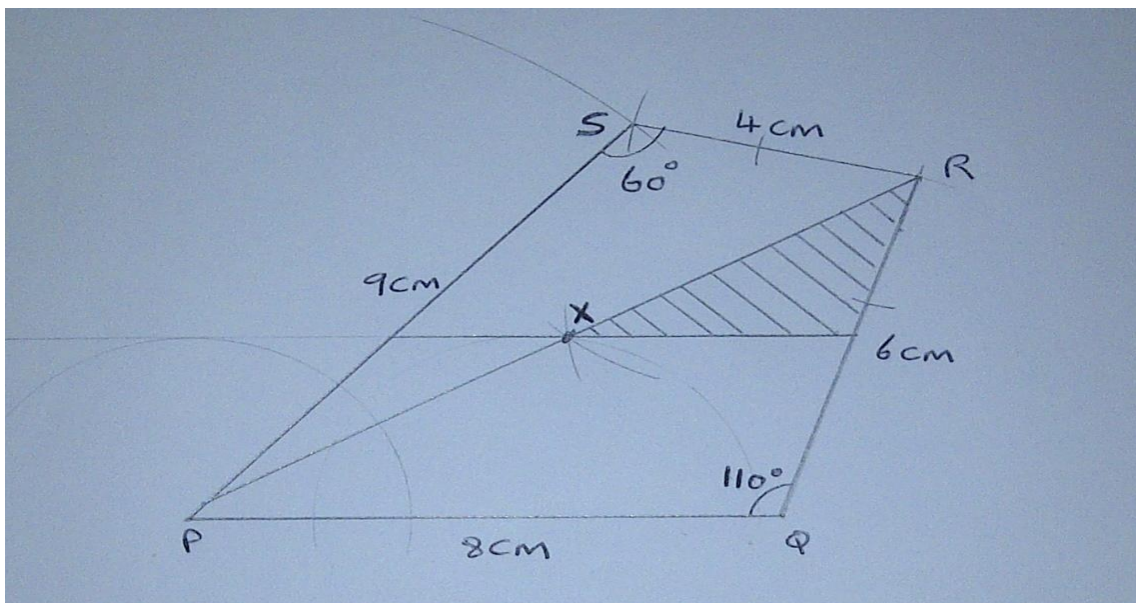   (i)   Equidistant from RS and RQ,
    (ii)   3cm from PQ.
(c) X is a point inside quadrilateral PQRS such that X is 3cm from PQ and equidistant from RS and RQ. Mark and label the point X.
(d) Another point Y, within quadrilateral PQRS, is such that it is equal to or more than 3 cm from PQ and nearer to RQ than RS. Shade clearly the region in which Y must lie.

Solution

Reasoning

(a) First draw the line PQ using a ruler. This should be 8 cm.

Centre at Q and measure 110° using a protractor. Put a dot at 110° before drawing the line QR.

Join the dot to point Q using a pencil drawing it as far as necessary. This line should be faint.

Measure 6cm using a compass, centre at Q and strike an arc such that it cuts the faint line.

Label the point R were the arc cuts the faint line. Make the line darker from Q to R.

Measure 9 cm using a compass, centre at P and strike an arc.

Measure 4cm, centre at R and strike an arc such that it intersects with the other arc. Label the point of intersection S.

Measure angle PSR using a protractor.

(b) (i) this is the angle bisector of angle SRQ

(ii) This is a line parallel to PQ such that the distance between the two lines is 3 cm.

Task

(a) (i) Construct a triangle ABC in which AC = 11.5cm, BC = 7cm and AB = 9.5cm.

(ii) Measure and write down the size of angle ABC.

(b) on the same diagram and within triangle ABC, construct the locus of points
   (i)   equidistant from AC and BC
   (ii)   7.5cm from A
   (iii)   2.5cm from AC

(c) given that N is a point within Triangle ABC such that it is greater than or equal to 7.5cm from A, nearer to AC than BC and less than or equal to 2.5cm from AC, shade clearly the region in which N must lie.

18. Sequences

Sequences involve arrangement of numbers in an orderly manner forming a pattern. In a sequence, there are terms: the first term (a_1), second term (a_2), third term (a_3) …………last term. The nth term is denoted by a_n, and it represents any term.

Let us consider the pattern below

$$9, 16, 25, 36, 49, ...$$

If you look closely at this pattern you will observe that these values are perfect squares of whole numbers with 9 as the first term. It can also be written as

$$3^2, 4^2, 5^2, 6^2, 7^2, ...$$

Here one can figure out that the 7th term will be $9^2 = 81$ and 8th term will be $10^2 = 100$. But what if one wants to find the 100th term, should one continue squaring the natural numbers up to the 100th term, surely this will be challenging and cumbersome. The easiest way to find any term of the sequence given its position is to derive the formula for the nth term. n represent any term, that is for first term(T_1) ,n = 1 and second term (T_2) ,n = 2.

For the sequence above

$$n_1 = 3^2 = (1 + 2)^2$$

$$n_2 = 4^2 = (2 + 2)^2$$

$$n_3 = 5^2 = (3 + 2)^2$$

Thus, **n^{th} term = $(n + 2)^2$**

18.1 Arithmetic progression (AP)

An AP is a sequence in which the difference between successive terms is constant. This difference is called the common difference (d).

For an AP: T_1 is the first term, T_2 is the second term and T_3 is the third term the common difference is given by

$$d = T_2 - T_1 \qquad or \qquad d = T_3 - T_2$$

Example

Consider the sequence 7, 9, 11, 13, . . .

$$T_2 - T_1 = 9 - 7 = 2$$

$$T_3 - T_2 = 13 - 11 = 2$$

Since $T_2 - T_1 = T_3 - T_2$ this sequence is an arithmetic progression. The pattern is , with 7 as the first term add 2 to the preceding term to get the next term. That is $T_5 = 15$, $T_6 = 17$ and $T_{11} = 27$.

The nth term of an AP is given by

$$\boxed{T_n = a + (n - 1)d}$$ **where a is first term and d is common difference**

For the sequence above, a = 7 and d = 2

The nth term = 7 + (n − 1)2

= 7 + 2n − 2

= 2n + 5

19. Statistics

This branch of mathematics deals with data representation and data summarization.

19.1 Data presentation

Data are raw facts or figures. In statistics graphs are used to summarize data and present it in meaningful ways. Data can be continuous or discontinuous. Continuous data cannot be categorized because it has a lot of intermediates between two values. Data on height or weight for example is continuous because it is difficult to categorize height or weight, some people are 1.5m others are 1.6m, 1.65m, 1. 68m and some people are 55kg, 55.5kg, 55.7kg etc. discontinuous data has few intermediates hence it can be categorized. Data on sex for example in discontinuous, because we can easily group males and female.

Depending on whether data is continuous or discontinuous, table, bar graph, histogram, line graph and pie chart are used to present data.
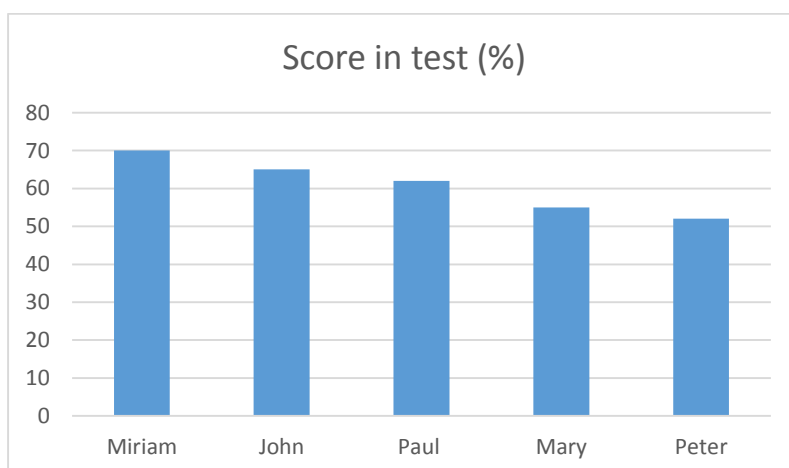
Tables

In a table data is presented in rows and columns. An example is shown below

| Name of pupil	Score in test (%)	Position in class
Miriam	70	1
John	65	2
Paul	62	3
Mary	55	4
Peter	52	5

Bar graphs.

These are used to represent discontinuous data using uniform bars that are uniformly spaced. For example comparing the number of males and females who attended a church service.

Histogram

Used to present continuous data using frequencies. In a histogram, rectangles are used to represent data. The areas of these rectangles are proportional to the frequencies.
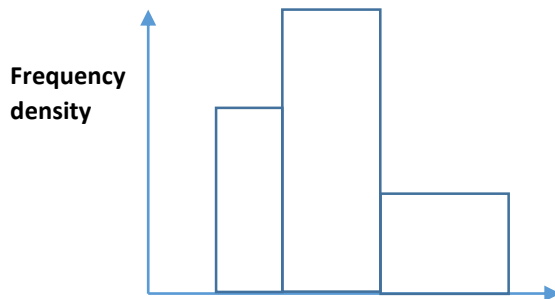
Pie charts

For fewer categories of discontinuous data, pie charts are used instead of bar graphs. Pie charts represent data using circles as the whole and sections of the circle to represent parts of the whole. Data can be presented in degrees with the total degree been 360° for complete revolution or in percentages.
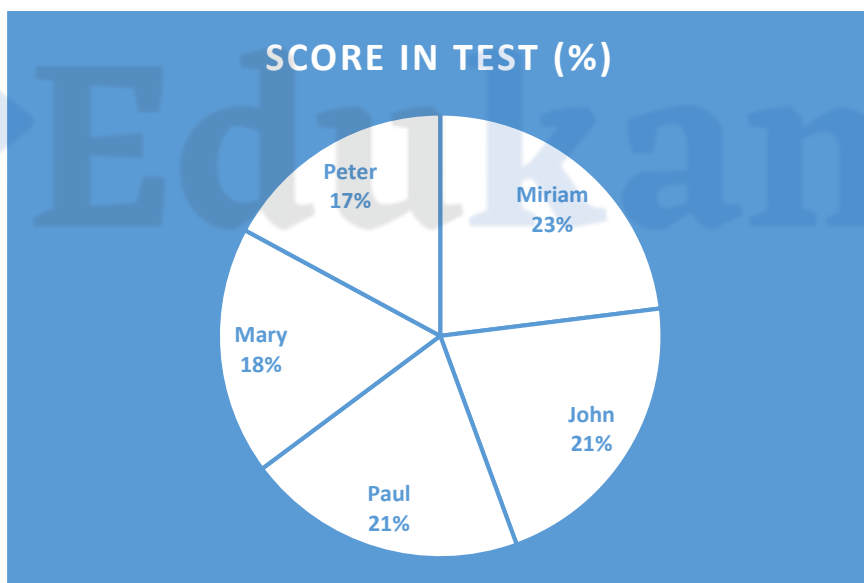
Line graph

In a line graph, continuous data are plotted and joined by a line.

19.2 Measures of central tendency

These measures are the mean, median and mode and are used to measure the average of a data set. A set of raw data is called a data set.

The following data are results of a class test out of 40. The class has 11 pupils.

20 25   10   30   10 15   30   8   12 10 10

The **mean** is the average of a data set. Its symbol is ($\bar{x}$). To find the mean add all the data values in a given then divide by the number of data values in that set. For the data set above

$$mean = \frac{20+25+10+30+10+15+30+8+12+10+10}{11}$$

$$= \frac{180}{11}$$

$$= \mathbf{16.4}$$

The **median** is the middle value when data has been arranged in ascending or descending order. For the data set above
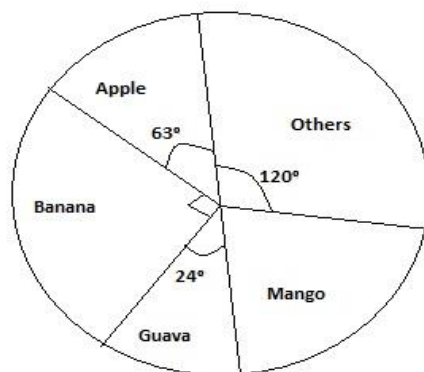
8 10 10   10   10 **12** 15 20 25 30   30

The median is 12.

The **mode** is the value with the highest frequency. The number that is occurring more than other numbers in a data set. It is possible to have two or more modes. For the data set above, the mode is 10.

Example

The pie chart below shows results of a survey conducted on 600 AIDS patients in one province on the fruit they preferred to eat.

(a) What fraction of patients chose banana?
(b) How many more chose mango than guava?

(a) The degree representing banana is 90°.

Fraction = $\frac{90°}{360°} = \frac{1}{4}$

(b) Let the x represent the degree for mango

x = 360° - 63° - 102° - 24° - 90° = 81

Patients who preferred mango = $\frac{81}{360}$ x 600 = 135

Patients who preferred guava = $\frac{24}{360}$ x 600 = 40

Patients who chose more mango than guava = 135 − 40 = 95

19.3 Frequency distribution

A frequency distribution table is one in which data values are arranged into intervals or classes along with the number of values (frequency) that fall in each class. A frequency distribution can be grouped or ungrouped.

For grouped frequency distribution, data values of a data set are grouped together in the classes. Data is grouped when there are too many different data values in a data set.

For a data set with few different values, data values are not grouped. Each value is in its own class. This is called ungrouped frequency distribution.

Ungrouped

Courses Taken	Frequency, f
1	25
2	38
3	217
4	1462
5	932
6	15

Grouped

Age of Voters	Frequency, f
18-30	202
31-42	508
43-54	620
55-66	413
67-78	158
78-90	32

19.4 Quartiles, Percentiles and Cumulative frequency graphs

For a given interval of data from the smallest to the largest value, the percentile gives information about how the data is spread. Percentile come from percentages.

Specific percentiles are called quartiles. There are three special quartiles:

1. lower quartile(Q_1) is the 25th percentile
2. median(Q_2) is the 50th percentile
3. upper quartile(Q_3) is the 75th percentile

Interquartile range (IQR) = $Q_3 - Q_1$

Example

The following table shows the total number of points scored by 110 candidates at Migule High School, in the 2007 grade 12 Final Examinations.

Points scored	$6 < x \le 8$	$8 < x \le 10$	$10 < x \le 12$	$12 < x \le 14$	$14 < x \le 16$	$16 < x \le 20$
Frequency	13	16	9	15	22	35

i.   Calculate the estimated mean points scored.   [2]
ii. Copy and complete the table showing the cumulative frequency distribution.   [2]

Points scored	≤ 6	≤ 8	≤ 10	≤ 12	≤ 14	≤ 16	≤ 20
Frequency	0	13					110

iii. Using a horizontal scale of 2cm to represent 2 points on the x – axis and a vertical scale of 2 cm to represent 10 candidates on the y – axis, draw a smooth cumulative frequency curve. [2]
iv.   Use your graph to find the median points scored.   [2]
v.   Find the interquartile range and the 55th percentile.
vi.   A scholarship was offered to two candidates. Find the probability that the two candidates chosen scored less than 14 points.   [2]
vii. Selection to the University of Zambia was done and only those who scored 14 points or less were selected. How many candidates were not selected from this group?   [1]

Solution

(a) To find the mean use the grouped frequency distribution table to construct the table below.

Class interval	Average of class interval (x)	Frequency (f)	fx
6 < x ≤ 8	7	13	91
8 < x ≤ 10	9	16	144
10 < x ≤ 12	11	9	99
12 < x ≤ 14	13	15	195
14 < x ≤ 16	15	22	330
16 < x ≤ 20	18	35	630
Total		∑f = 110	∑fx =1489

Therefore mean $(x) = \frac{\sum x}{\sum f}$ **the symbol ∑ means summation (total)**

$= \frac{1489}{110}$

$= 13.5$

(b) To complete the cumulative frequency distribution table (CFDT), use the frequency of the grouped frequency distribution table (GFDT). Since the first value of frequency for (CFDT) is 0 add this to the first frequency of (GFDT) to get the next value. 0 + 13 = 13. To the sum add the following frequency of (GFDT) to get the next value, 13 + 16 = 29. Keep on doing this until all the boxes have been filled. i.e. 29 + 9 = 38, 38 + 15 = 53, 53 + 22 = 75 and 75 + 35 = 110.

Points scored	≤ 6	≤ 8	≤ 10	≤ 12	≤ 14	≤16	≤ 20
Frequency	0	13	29	38	53	75	110

(c) Use the cumulative frequency distribution table to draw the cumulative frequency curve. Points scored should be on the x- axis and frequency (number of candidates) on the y – axis.
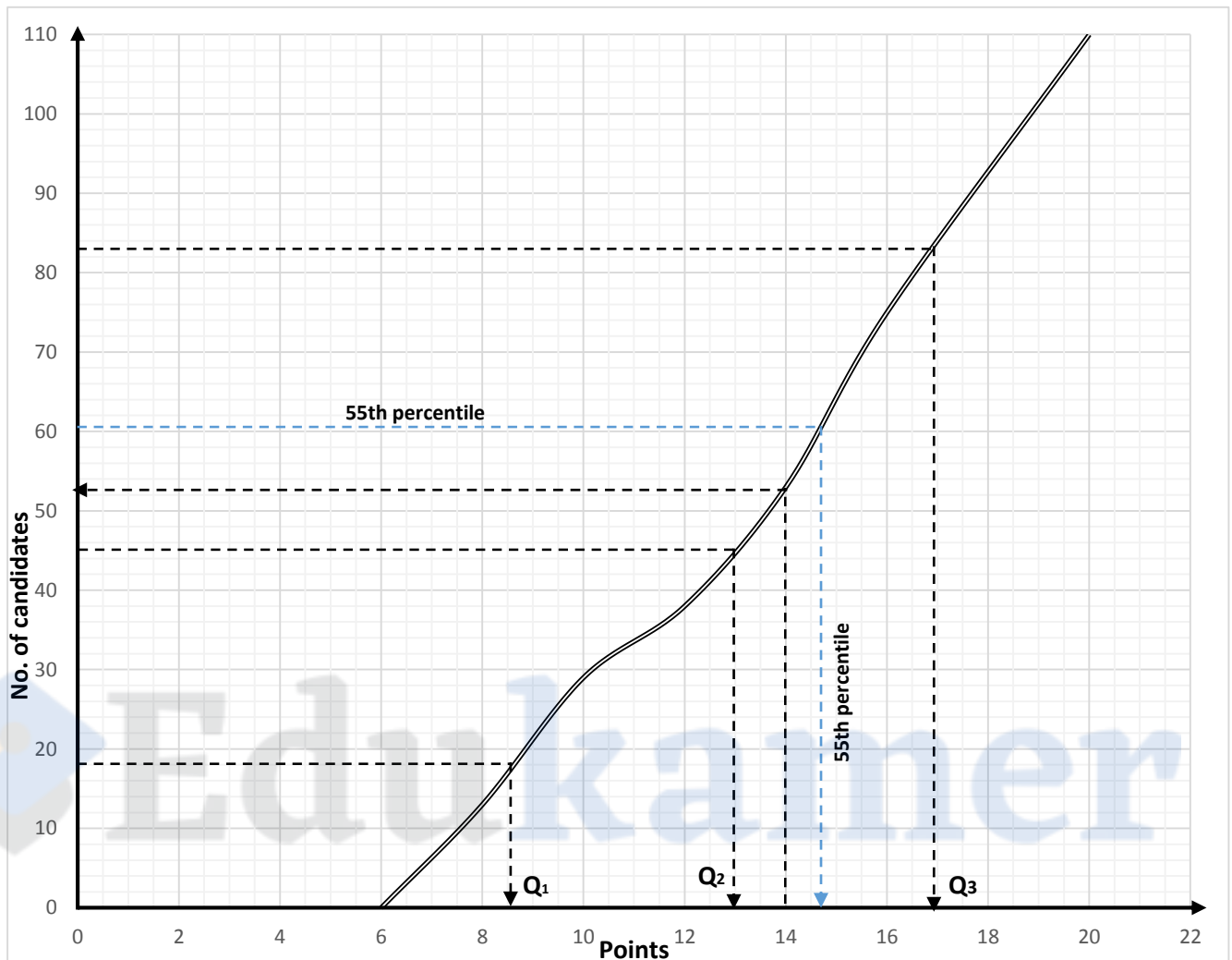
(d) To find median point multiply the total number of candidates by $\frac{1}{2}$. i.e. $\frac{1}{2}$ x 110 = 55. On the y – axis, draw a horizontal line at 55 (y = 55) until you just touch the curve. At the point you touch the curve draw a vertical line towards the negative x - axis until you touch the x – axis. The x – value at the point you touch the x – axis is the median. on the graph the median (Q_2) = 13

(e) First find upper quartile (Q_3). Unlike for the mean multiply the total of candidates by $\frac{3}{4}$. i.e. $\frac{3}{4}$ x 110 = 82. 5. On the y – axis go at y = 82.5 and follow the steps as for the median above. Q_3 = 17

Now find the lower quartile. $\frac{1}{4}$ x 110 = 27.5. Go at y = 27.5 and follow the steps as for the median above. Q_1 = 8.5

Finally, interquartile range = Q3 – Q1 = 17 – 8.5 = 8.5

To find the 55th percentile, divided the percentage by 100 and multiply the total number of candidates. i.e. $\frac{55}{100}$ x 110 = 60.5. On the y – axis go at y = 60.5 and follow the steps like we did with quartiles. The 55th percentile = 14.75.

On the point axis (x – axis) go at 14 points (x = 14) draw a vertical line till you touch the curve. Get the y – value at the point you touch the curve. 52 candidates were selected.

Task

The table below shows the masses of 100 babies at birth, recorded at a hospital.

Mass (x kg)	1.5 < x ≤ 2.0	2.0 < x ≤ 2.5	2.5 < x ≤ 3.0	3.0 < x ≤ 3.5	3.5 < x ≤ 4.0	4.0 < x ≤ 4.5	4.5 < x ≤ 5.0
No. of babies	3	12	20	24	25	14	2

(a) Copy and complete the cumulative frequency table below.   [2]

Points scored	≤ 1.5	≤ 2.0	≤ 2.5	≤ 3.0	≤ 3.5	≤ 4.0	≤ 4.5	≤ 5.0
Frequency	0	3	15					100

(b) Using a horizontal scale of 2cm to represent 0.5kg for masses from 1.5kg to 5.0kg and a vertical scale of 2cm to represent 10 babies, draw a smooth cumulative frequency curve. [3]
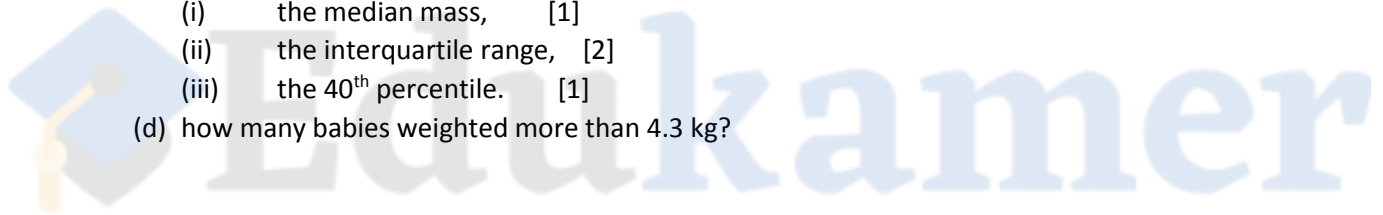
(c) showing your method clearly, use your graph to graph to estimate
   (i)   the median mass,   [1]
   (ii)   the interquartile range,   [2]
   (iii)   the 40th percentile.   [1]

(d) how many babies weighted more than 4.3 kg?

20. Kinematics

Kinematics is the study of motion. The main focus of this chapter is on distance and speed and their graphs.

Distance is the length between two points. It is measured in metre (m). For very long distances the kilometre (km) is used. Speed is the rate of change of distance. Speed is measured in metre per second (m/s). Kilometre per hour (km/h) is also used.

$$speed = \frac{distance}{time}$$

20.1 Other important terms

(i) Velocity: is the rate of change of displacement with time. It is measured in metre per second (m/s). Unlike speed has magnitude (size) only, velocity has both size and direction.

(ii) Acceleration: is the rate of change of velocity. It is measured in metre per second squared (m/s^2).

$$acceleration = \frac{change\ in\ velocity}{time\ taken}$$

$$a = \frac{V - u}{t}$$

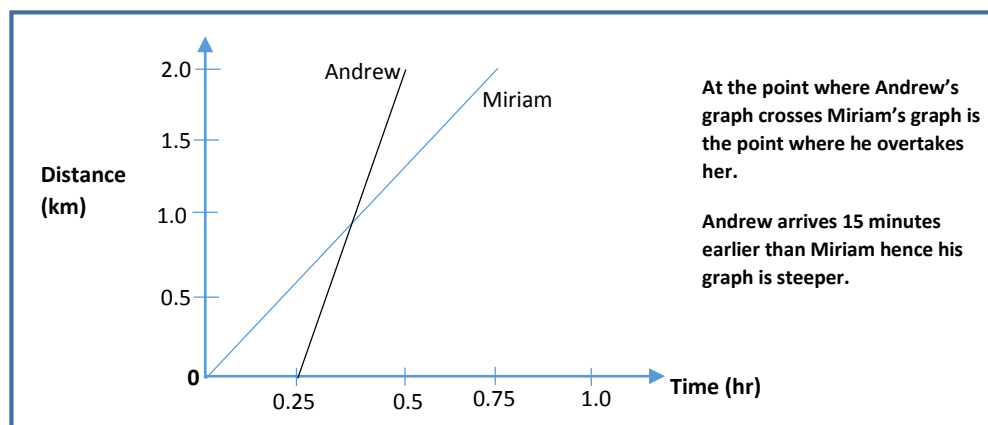Where V is final velocity and u is initial velocity

(iii) Retardation: is negative acceleration due to decreasing velocity.

20.2 Graphical representation of motion
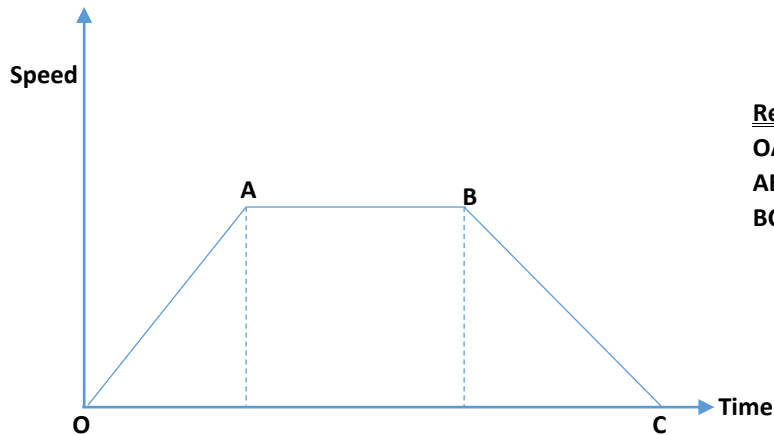
(i) **Distance time graph**

This shows the relationship between distance and time. Distance is on the y – axis and time on the x – axis.

Example. **One a particular Miriam decides to walks to school to attend classes. The school is 2 km from her home and it takes her 45 minutes to cover this distance. Andrew decides to take a bus 15 minutes after Miriam has left. It takes him 15 minutes to cover the same distance. Represent this information on a graph.**

At the point where Andrew's graph crosses Miriam's graph is the point where he overtakes her.

Andrew arrives 15 minutes earlier than Miriam hence his graph is steeper.

(ii) Speed time graph

This graph shows the relationship between speed and time.

Region
OA – increasing speed (acceleration)
AB – constant speed (no acceleration)
BC – decreasing speed (retardation)

To find the distance moved in a certain period of time, calculate the area under the graph. For example, the graph above can be split into two triangles and a rectangle. The areas of these shapes can be calculated and added to give the total distance covered from O to C.

Example 1

The figure below shows how a car is retarded uniformly from a speed of 36 m/s to a speed of 24 m/s in a time of 40 seconds and then uniformly brought to rest after a further 10 seconds.
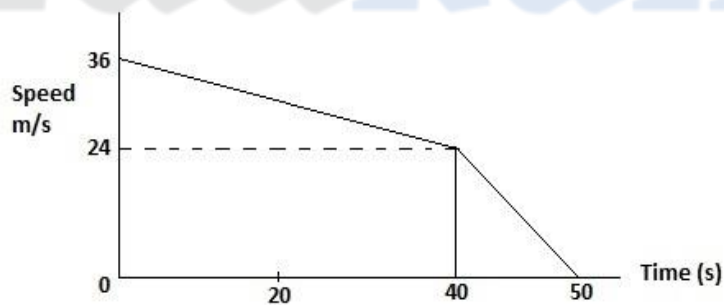
Calculate:

(a) the retardation of the car during the first 40 seconds.
(b) the distance travelled in the first 40 seconds.

Solution

(a) $a = \dfrac{v-u}{t}$

$= \dfrac{24\frac{m}{s} - \frac{36m}{s}}{40s}$

$= -0.3 \ m/s^2$

The negative sign signifies retardation

(b) D = area of Δ + area of rectangle

$= (\frac{1}{2} \times 40 \times 12) + (40 \times 24)$

$= 1200m$

Example 2

The diagram below shows the speed – time graphs of two trains leaving a station at the same time. Train A accelerates uniformly from rest to a speed of 40 m/s in 4 seconds. It moves at this speed for 6 seconds before coming to rest in a further 2 seconds. Train B accelerates uniformly from rest to a speed of v m/s in 6 seconds then comes to rest in a further 6 seconds.
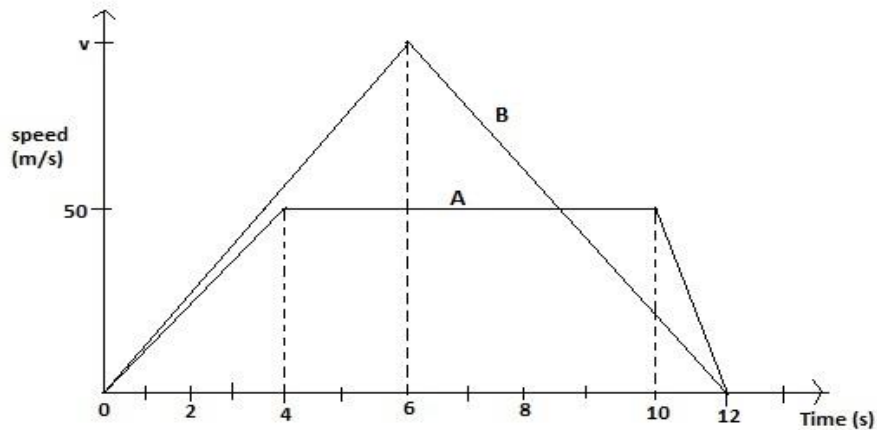
(a) Given that the trains moved the same distance, calculate the maximum speed, v, of train B.

(b) Find the distance between the trains during the first 6 seconds.

Solution

(a) find the distances for each train and equate them

Distance by train A = ($\frac{1}{2}$ x 4 x 50) + (6 x 50) + ($\frac{1}{2}$ x 2 x 50)

$\qquad\qquad$ = 100m + 300m + 50m

$\qquad\qquad$ = 450m

Distance by train B = $\frac{1}{2}$ (12 x v)

$\qquad\qquad$ = 6v

Since the trains travelled the same distance, equate the distances and solve for v

$\qquad$ 6v = 450m

Divide by 6 on both sides

$\qquad$ v = 75m

(b) Train A.

Distance = ($\frac{1}{2}$ x 4 x 50) + (50 x 2) = 200m

Train B.

Distance = $\frac{1}{2}$ x 6 x 75 = 225m

Therefore the distance between the two trains = 250m – 200m = 50m

Task

The diagram below shows the speed – time graphs of a van and a car. The van, starting from rest, accelerates uniformly for 8 seconds until it reaches a speed of 6m/s. it then continues to travel at this constant speed.
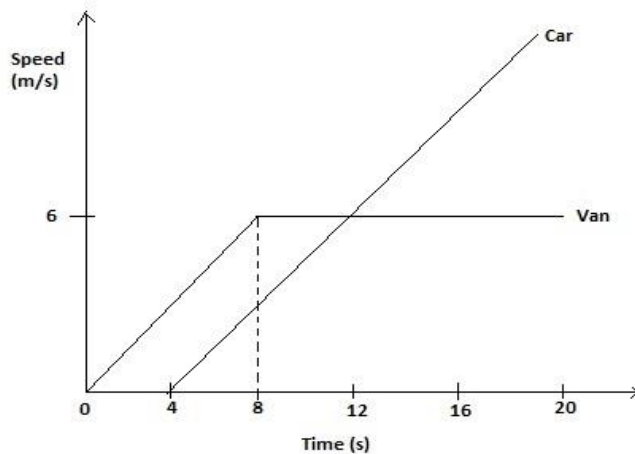
(i)   find the acceleration of the van during the first 8 seconds.
(ii)   find the distance travelled by the van in 20 seconds.
(iii)   the car starts from the same place as the van, but 4 seconds later, and accelerates uniformly until it overtakes the van as shown on the diagram above. Given that overtaking occurs when the van has been travelling for 20 seconds, calculate the speed of the car at this instant.

21. Transformation

Geometric shapes can be represented by a pair of coordinates of the xoy plane. The original shape called the object can be moved (mapped) to a new position following a certain set of rule. This is called transformation. The transformed object is called the image. The image may or may not be the same size as the object. Objects can be transformed under six (6) different types of transformation: translation, reflection, rotation, enlargement, sheer and stretch.

21.1 Types of transformation

(i)   translation

A translation is fully describe by the translation vector matrix of the type $\begin{bmatrix} x \\ y \end{bmatrix}$. x is the number of times the objective is moved either in the positive or negative x – axis direction. y is the number of times the object is moved in the positive or negative y – axis direction. In a transition, each point on the object moves the same number of times in the x – axis and y – axis as other points on the object. Under a translation the shape and size of the object is unchanged. The image faces the same direction as the object.
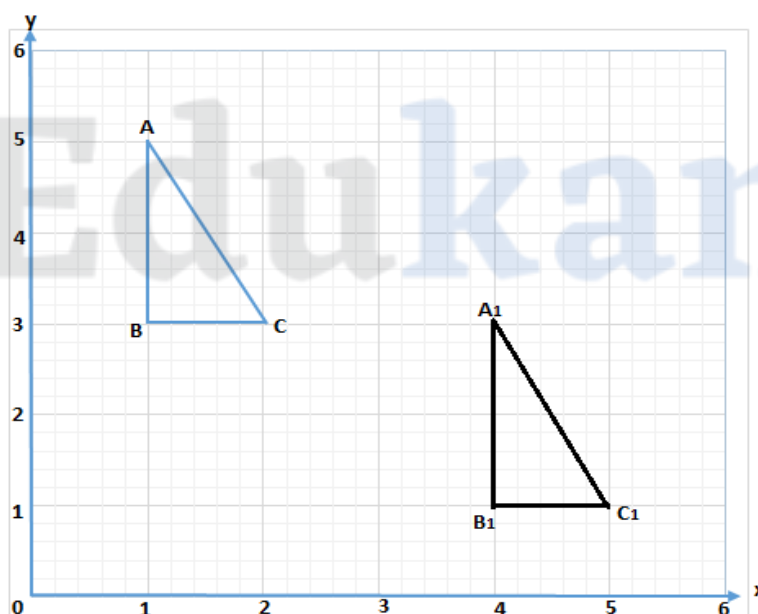
ΔABC has been mapped to ΔA$_1$B$_1$C$_1$ by translation with translation vector matrix $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

In other words every point on ΔABC has been moved 3 times toward the positive x – axis then 2 times towards the negative y – axis (downwards) to produce ΔA$_1$B$_1$C$_1$.
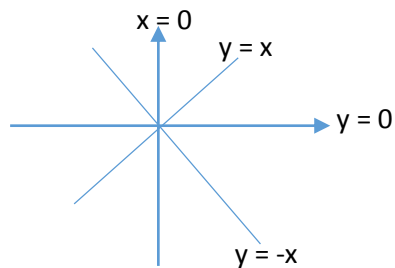
(ii)   **reflection**

In a reflection the object is reflected in the line called the mirror line to form the image. To describe a reflection, state the equation of the mirror line.
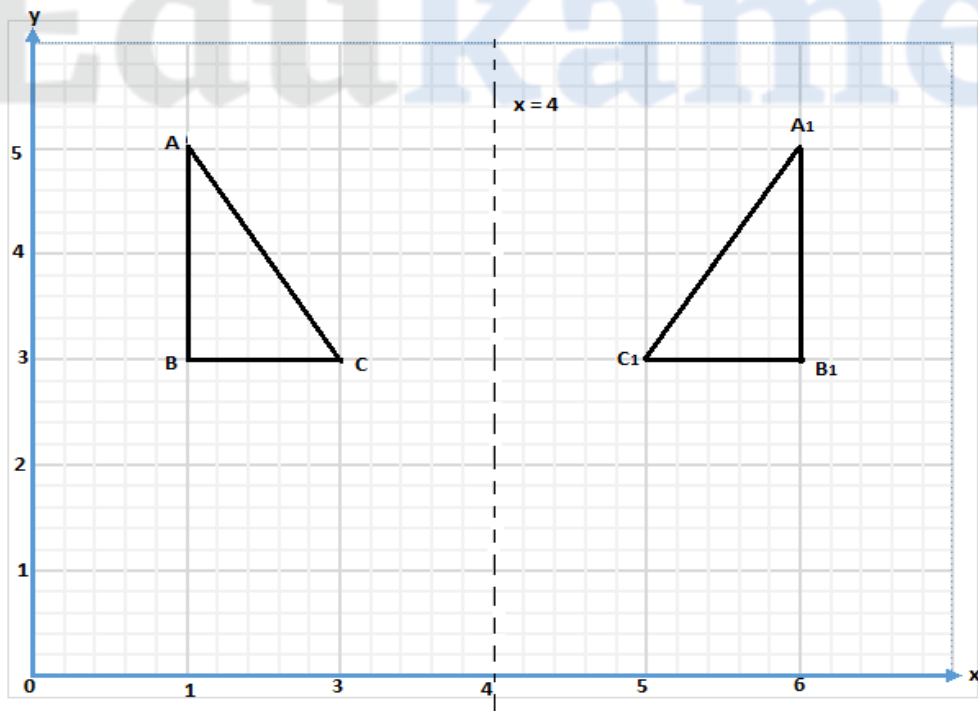
Properties of the image

- Image is the same size as object
- The image and the object are on opposite sides of the mirror line.
- It is literally inverted (back to front)
- The distance from the object to the mirror line is equal to the distance of the image from the mirror line

Equations of common mirror lines

### Example

$\triangle ABC$ has been mapped to $\triangle A_1 B_1 C_1$ under a reflection in the mirror line $x = 4$. Note that the image is literally inverted and same size as object. The mirror line is shown as a dotted line.

(iii) **rotation**

Under a rotation the object is rotated either clockwise or anticlockwise to a new position. A rotation is fully described by stating the direction of rotation (clockwise or anticlockwise), angle of rotation and the centre of rotation.

Under a rotation, the shape and size of the object is unchanged.
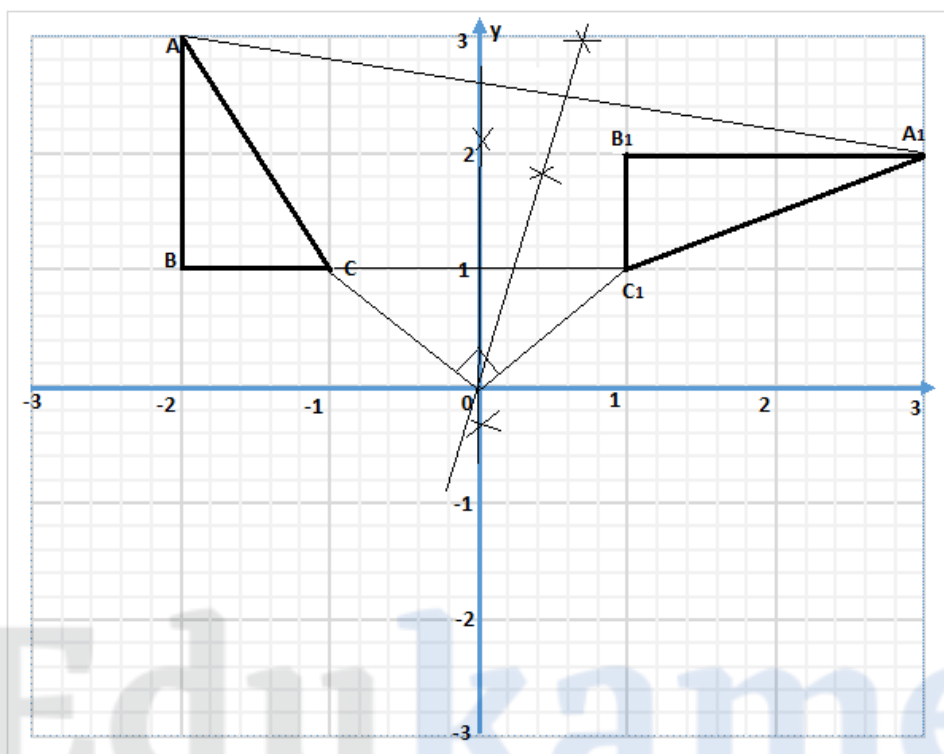
ΔABC has been mapped to $\Delta A_1B_1C_1$ under a rotation. To find the centre of rotation, join two corresponding points by a straight line (e.g A and A1), bisect this line using a compass and pencil drawing the straight line as far as possible. Bisect another line joining other corresponding points (e.g C and C1), where the bisecting lines meet is the centre of rotation. In the example above, the centre is (0, 0).

To find the angle of rotation, join a point on the object (e.g C) to the centre and a corresponding point on the image (e.g) to the centre. Measure the angle between the two lines using a protractor. In this case the angle of rotation is 90° clockwise.

In all, ΔABC has been mapped to $\Delta A_1B_1C_1$ under a rotation, with centre of rotation (0, 0) and angle of rotation of 90° clockwise.

(iv) **enlargement**

Under an enlargement the object is either enlarged or reduced in size. An enlargement is fully described by stating the centre of enlargement and the scale factor (k).

$$k = \frac{distance\ of\ a\ point\ on\ image\ from\ centre}{distance\ of\ corresponding\ point\ on\ object\ from\ centre}$$

Example
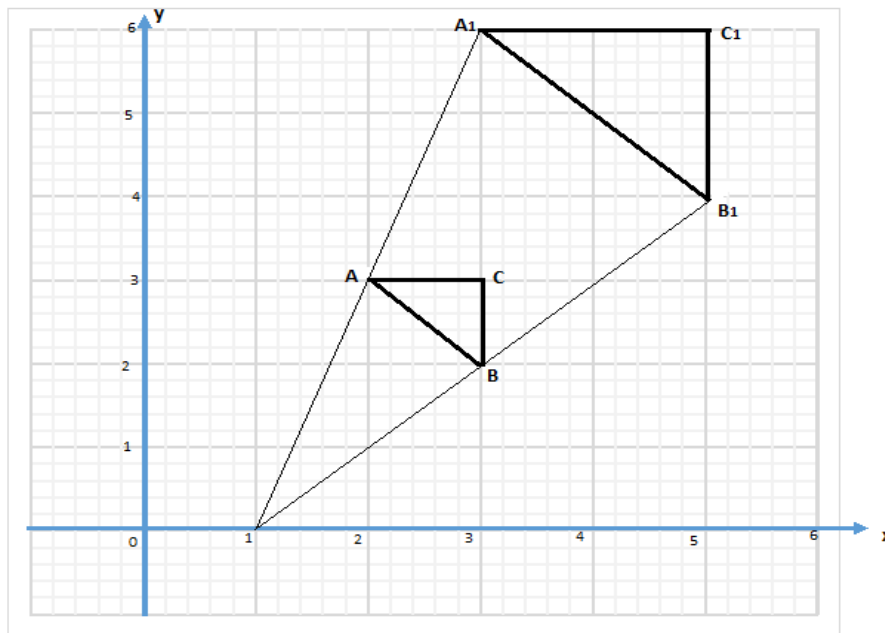
To find the centre of enlargement join two corresponding points on object and image (e.g. A and A_1) drawing them as far as possible. Join another pair of corresponding points (e.g. B and B1) drawing the line as far as possible. Where these two extended lines meet is the centre. In the example above, the centre is (1, 0).

$$\text{Scale factor} = \frac{\text{distance from centre to A1}}{\text{distance from centre to A}} = \frac{6}{3} = 2$$

Example 2
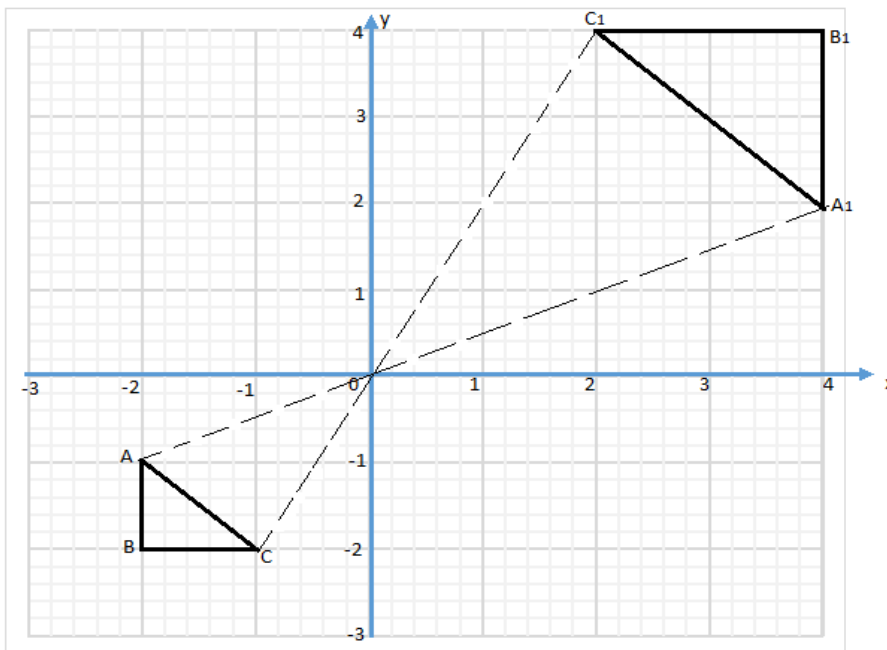
Here the centre is (0, 0) and the scale factor is -2. That is $k = \frac{OC1}{OC} = \frac{4}{-2} = -2$. The scale factor is negative because the object and the image are on opposite sides of the centre of enlargement.

(v)   **shear**

A shear is fully described by stating the invariant line and the shear factor (k). An invariant line is a fixed line on which points lying on it does not move under a transformation. The shape of the object changes under a shear but the area is the unchanged. That is area of the object = area of image. The common invariant lines are the y – axis (x = 0) and x – axis (y = 0).

$$\text{Shear factor (k)} = \frac{\text{distance moved by a point}}{\text{perpendicular distance of point from the invariant line}}$$

Points on the invariant lines (x-axis and y – axis)

When the x –axis is the invariant line, the point(s) on it will not move but other points will move parallel to it (horizontally). If the y – axis is the invariant line and some points on the object are lying on it, those points will not move, but other point(s) will move parallel to it (vertically).
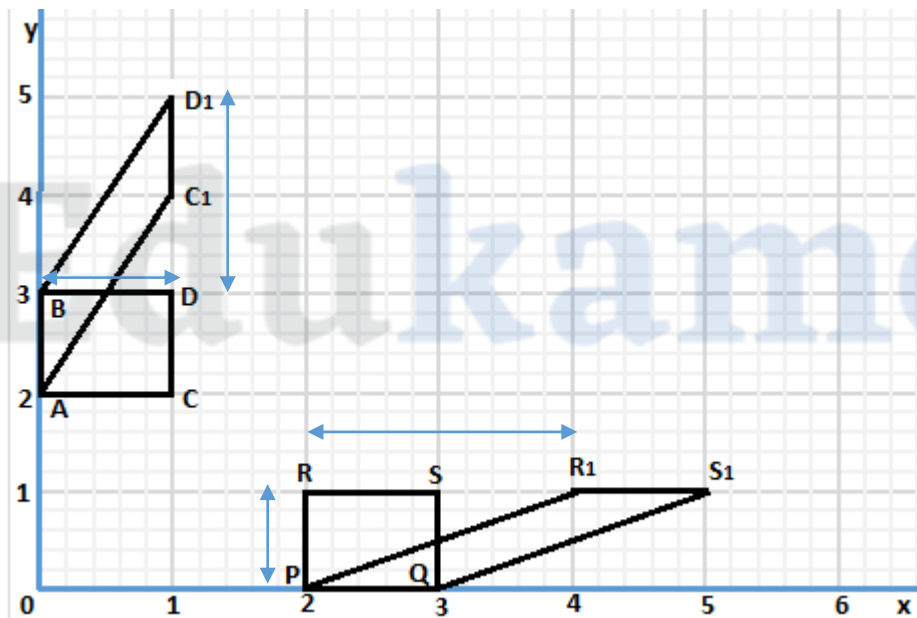
For square ABCD, invariant line is y – axis. The points C and D have moved parallel to the y – axis to C_1 and D_1. Note the points A and B are on the invariant line and they have not moved.

$$\text{Shear factor (k)} = \frac{\text{distance from D to } D_1}{\text{perpendicular distance of point D from the invariant line}} = \frac{2}{1} = 2$$

For square RSPQ, invariant line is x – axis. The points R and S have moved parallel to the x – axis to R_1 and S_1. Note the points P and Q are on the invariant line and they have not moved.
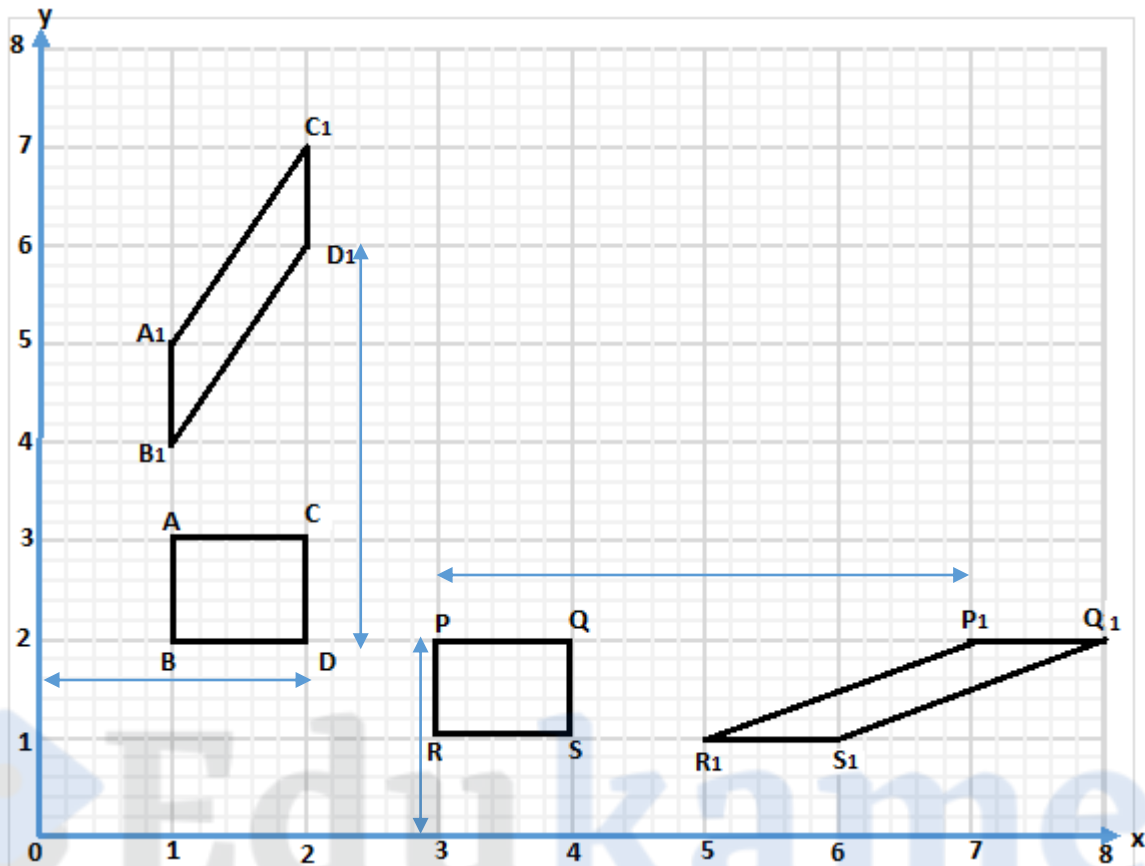
$$\text{Shear factor (k)} = \frac{\text{distance from R to } R_1}{\text{perpendicular distance of point D from the invariant line}} = \frac{2}{1} = 2$$

Here the points are not on the invariant line so all the points on the object move.

For square ABCD, invariant line is y – axis. Since there are no points lying on the invariant line, all point of the object have moved.

Shear factor (k) = $\dfrac{\text{distance from D to } D_1}{\text{perpendicular distance of point D from the invariant line}} = \dfrac{4}{2} = 2$

For square PQRS, invariant line is x – axis. Since no points are lying on the invariant line, all the points on the object have moved.

Shear factor (k) = $\dfrac{\text{distance from P to } P_1}{\text{perpendicular distance of point P from the invariant line}} = \dfrac{4}{2} = 2$
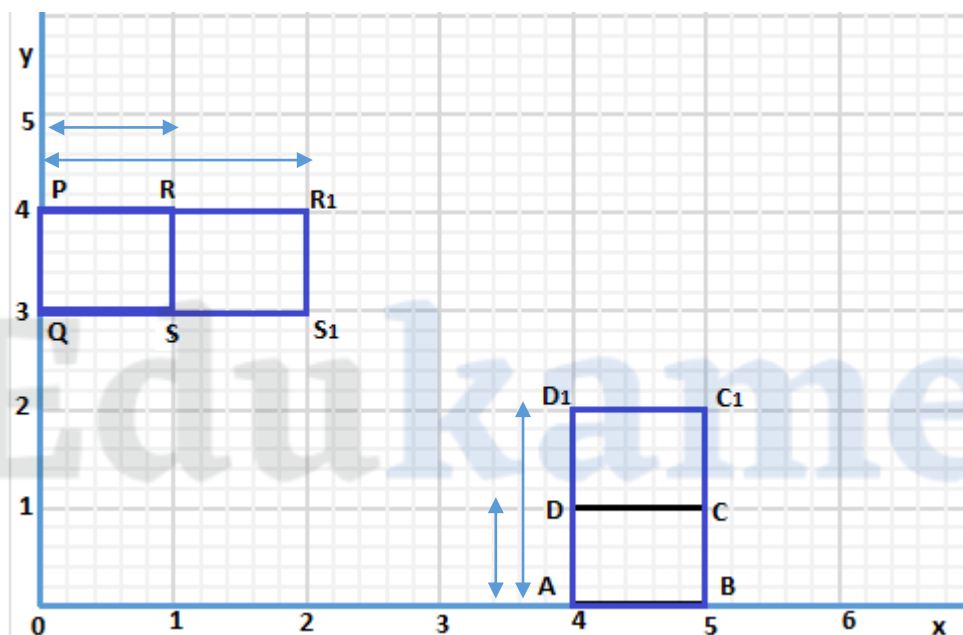
(vi)   stretch

In a stretch the size of the object is either enlarged or reduced in one direction only. A stretch is fully described by a stretch factor and an invariant line. Invariant points are points which do not move under a transformation. The most common invariant lines are the y – axis and x – axis.

$$\text{Stretch factor (k)} = \frac{\text{distance moved by a point from invariant line}}{\text{distance of point from the invariant line}}$$

Points on invariant line (x – axis and y – axis)

Note that the points on the invariant line do not move.

For the square PRQS, the invariant line is the y- axis. The points R and S have moved perpendicular to the y – axis to R_1 and S_1. Note again that like for the shear, points on the invariant line in a stretch do not move, P and Q have not moved.

Stretch factor (k) = $\frac{\text{distance moved by point R from invariant line}}{\text{distance of point R from the invariant line}} = \frac{2}{1} = 2$
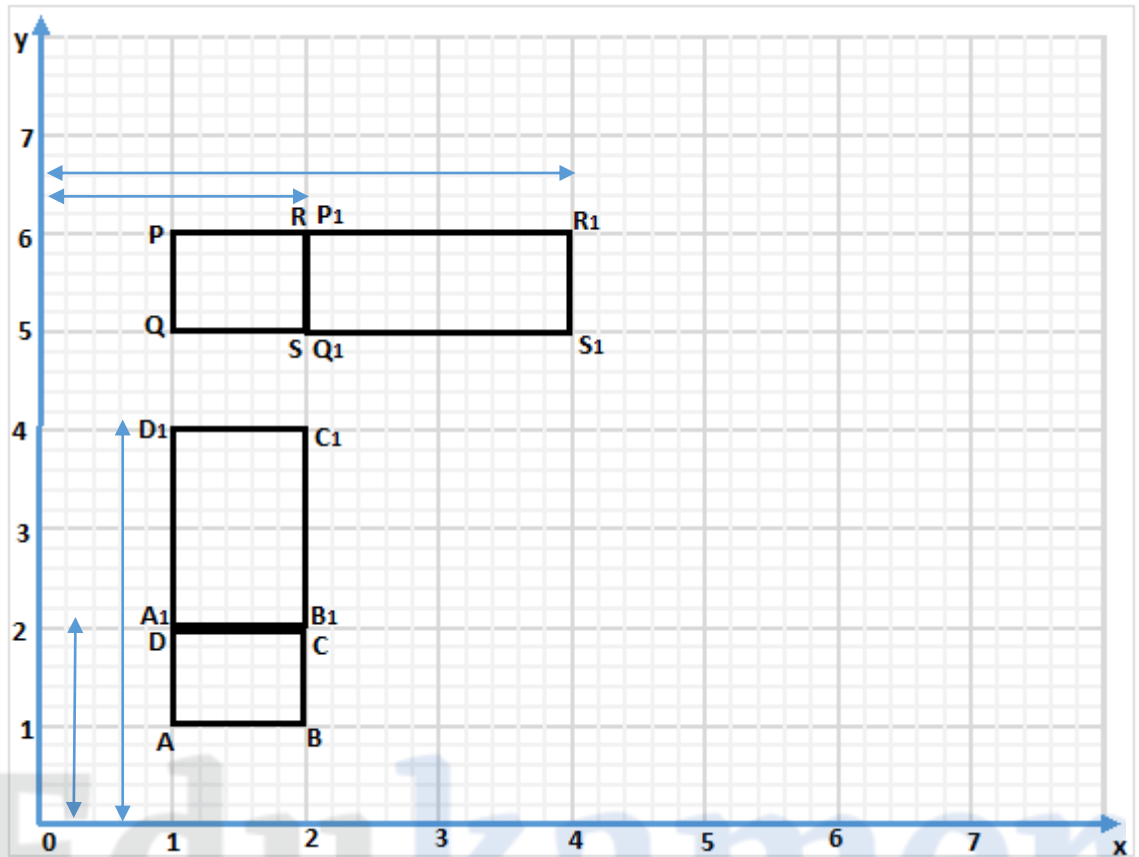
For the square ABCD, the invariant line is the x- axis. The points D and C have moved perpendicular to the x – axis to C_1 and D_1. The points A and B have not moved.

Stretch factor (k) = $\frac{\text{distance moved by point D from invariant line}}{\text{distance of point D from the invariant line}} = \frac{2}{1} = 2$

Points not on invariant line (x –axis and y – axis)

For the square PRQS, the invariant line is the y- axis. Since there are no points on the invariant line, all the points on the object have moved perpendicular to the invariant line.

$$\text{Stretch factor (k)} = \frac{\text{distance moved by point R from invariant line}}{\text{distance of point R from the invariant line}} = \frac{4}{2} = 2$$

For the square ABCD, the invariant line is the x- axis. All the points on the object have moved perpendicular to the invariant line.

$$\text{Stretch factor (k)} = \frac{\text{distance moved by point D from invariant line}}{\text{distance of point D from the invariant line}} = \frac{4}{2} = 2$$

21.2 Transformation by using matrices

It is easy to find the coordinate(s) of the image given the coordinate(s) of the object and the specific matrix under which it was transformed. The formula below can be used:

$$\text{Image} = \text{Matrix X object}$$

The table below shows some commonly used matrices.

Reflection

Mirror line	Matrix	Mirror line	Matrix
x – axis (y = 0)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	y = x	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
y – axis (x = 0)	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	y = -x	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Rotation

Rotation with centre (0, 0)

90° clockwise	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	90° anticlockwise	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
180° (either direction)	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$		

Enlargement

Scale factor k and centre (0, 0)

Scale factor k Centre (0, 0)	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Shear

x– axis invariant (shear factor k)	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	y – axis invariant (shear factor k)	$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

Stretch

| x– axis invariant (stretch factor k)	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$	y – axis invariant (stretch factor k)	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

Example

Using a scale of 2cm to represent 1 unit on each axis, draw x and y axes for -3 ≤ x ≤ 7 and -3 ≤ y ≤ 7.

(a) ΔABC has vertices A(3, 6), B(3, 7) and C(1, 7). $\Delta A_1B_1C_1$ has vertices A_1(-1, 4), B_1(-2, 4) and C_1(-2, 2).
  (i) draw and label ΔABC and $\Delta A_1B_1C_1$.
  (ii) describe fully a single transformation which maps ΔABC onto $\Delta A_1B_1C_1$.

(b) The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ maps ΔABC onto $\Delta A_2B_2C_2$.

  (i) find the coordinates of $\Delta A_2B_2C_2$
  (ii) draw and label $\Delta A_2B_2C_2$.

(c) $\Delta A_1B_1C_1$ can be mapped onto $\Delta A_3B_3C_3$ by a translation. Given that A_3 is the point (3, -1),
  (i) find the column vector representing this translation,
  (ii) draw and label $\Delta A_3B_3C_3$

(d) $\Delta A_4B_4C_4$ has vertices A_4(0, 3), B_4(0, 1) and C_4(4, 1).
  (i) Draw and label $\Delta A_4B_4C_4$.
  (ii) Describe fully the transformation which maps ΔABC onto $\Delta A_4B_4C_4$.

Solution

(a)
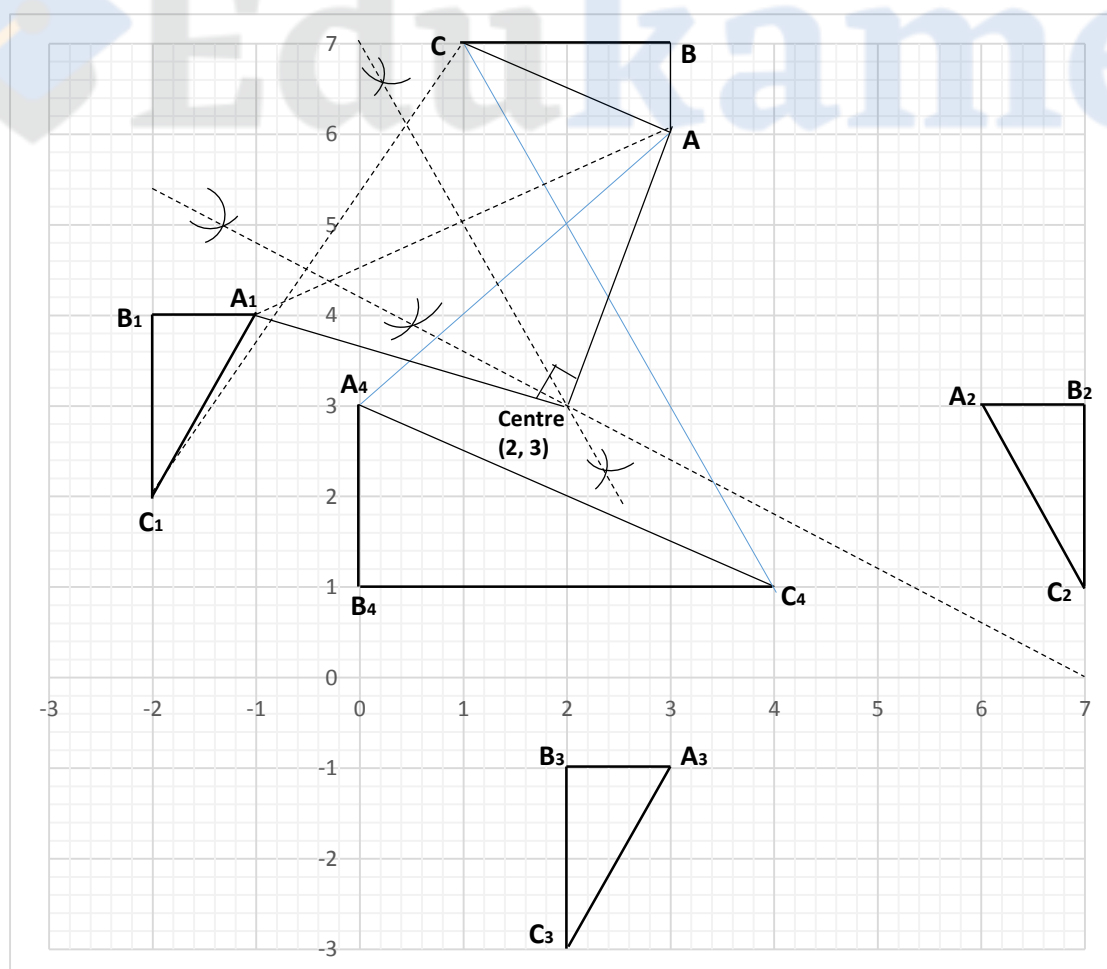
ii. It is a rotation in the anti-clockwise direction with centre (2, 3).

(Reasoning: to find the centre of rotation join at least to corresponding sides i.e. A to A1 and C to C1. Construct a perpendicular bisectors of lines AA1 and CC1. Where these two bisectors meet is the centre. To find angle of rotation join A to the centre and A1 to the centre, then measure the angle at the centre using a protractor.

(b) (i) Image = matrix X object

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 & 1 \\ 6 & 7 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 7 & 7 \\ 3 & 3 & 1 \end{pmatrix}$$

Thus: A_2 (6, 3), B_2 (7, 3) and C_2 (7, 1)
(ii) on the graph

(c) (i) A_1 (-1, 4) is the object and A_3 (3, -1) is the image. To get to A_3 from A_1 we have to move 4 times towards the positive x – axis and 5 times towards the negative y – axis(downwards). Therefore the translation vector matrix is $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$.

(ii) on the graph. To draw the $\Delta A_3B_3C_3$, move the other points B1 and C_1 like A_1 was moved then join the three points.

(d) (i) on the graph

(ii) since the size of the shape has increased it is an enlargement. To find the centre, join two corresponding points A to A_4 and C to C_4. Where these two lines meet is the centre. Here the centre is (2, 5). Now find the scale factor (k).

$$\text{scale factor (k)} = \frac{\text{distance of A4 from centre}}{\text{distance of A from centre}} = \frac{2}{-1} = -2$$

Therefore, the transformation which maps ΔABC onto $\Delta A_4B_4C_4$ is an enlargement with centre (2, 5) and scale factor (k) -2.

22. Probability

The probability of an even A denoted as P (A) is the chance or likelihood of the event happening. There is no number in probability that is greater than 1 or less than 0. 1 is the maximum number and it represent a certainty. 0 is a minimum and it represent an absolute impossibility.

$$P (A) = \frac{\text{number of favourable outcomes}}{\text{number of all possible outcome}}$$

Example.

John has 4 marbles in his pocket, 1 red, 1 blue and 2 green. He decides to pick 1 marble at random. Find the probability that the marble is

(a) Red
(b) Blue
(c) Green

Solution

(a) P(red) $= \frac{1}{4}$ no. of red marbles 1, total no. of marbles 4

(b) P(blue) $= \frac{1}{4}$

(c) P(Green) $= \frac{2}{4} = \frac{1}{4}$

22.1 Dependent event

Some events are affected by other events. If the probability of an event happening affects the probability of another event happening, the event is called dependent. Using the example above, if john picked the red marble without replacing it, the probabilities of picking other marbles are affected. The probability of picking a blue marble is $\frac{1}{3}$ and the probability of picking the green marble is $\frac{2}{3}$. Note that when a marble is removed without replacement, the total number of marbles is affected.

22.2 Independent event

If the probability of an event happening does not affect the probability of another event happening, the event is said to be independent. With reference to the example above, if john picks the red marble and replaces it, the probabilities of picking other marbles in not affected.

22.3 Conditional probability

Conditional probability is concerned with the probability of say event *B* occurring, given that event *A* has already taken place.

If *A* and *B* are independent events, then the fact that event *A* has already occurred will not affect the probability of event *B*.
If *A* and *B* are dependent events, then event *A* having occurred will affect the probability of event *B*.

22.4 Mutually exclusive events

These are events which cannot occur together. If event A and event B are mutually exclusive then when event A has happened, the probability of event B is 0.
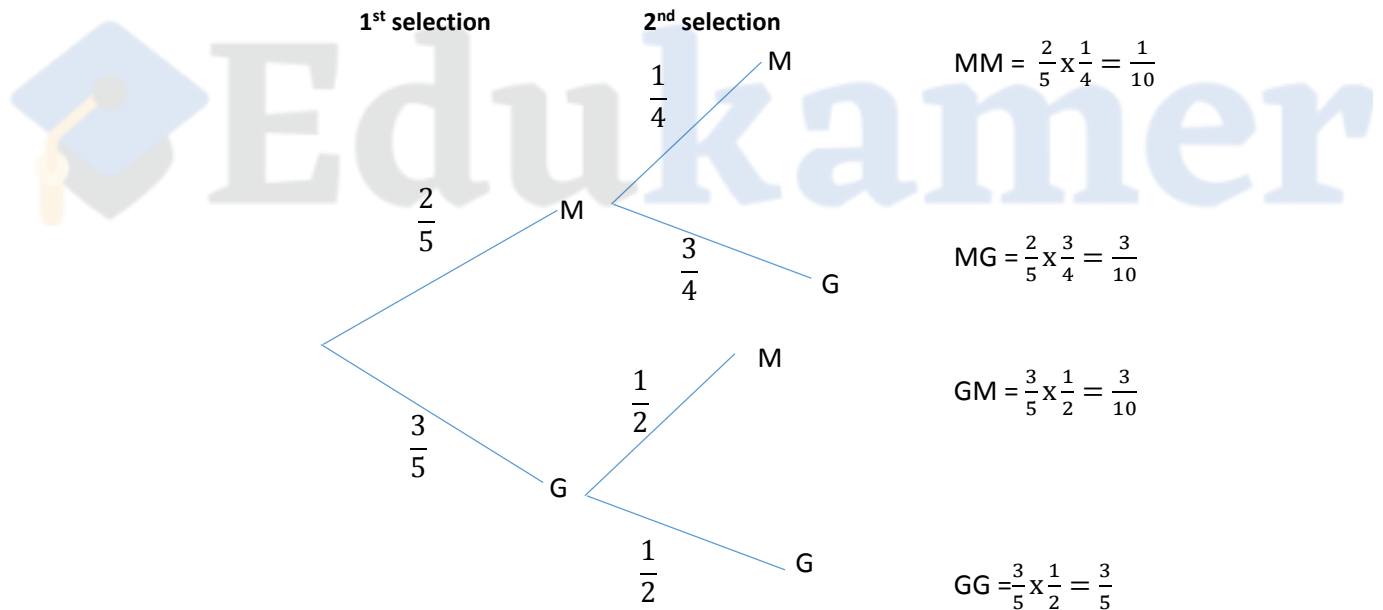
22.5 Laws of probability

(i) Addition. If the probabilities are joined by the word 'or' add. For example, the probability of event A or event B happening is P (A) + P (B) provided that these events are mutually exclusive.

(ii) Multiplication. If probabilities are joined by the word 'and' multiply. For example, the probability of event A and event B happening is P (A) x P (B).

22.6 The tree diagram

This is the diagrammatic representation of probabilities.
A box contains 5 apple, 3 are green and 2 are maroon. Construct a tree diagram when an apple is picked at random from the box without replacement, then another apple is picked.

1st selection   **2nd selection**

$$MM = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$$

$$MG = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$$

$$GM = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$$

$$GG = \frac{3}{5} \times \frac{1}{2} = \frac{3}{5}$$

The letters G and M represent green and maroon respectively. Since there are only 2 selection the tree has 2 branches, 1 for green and 1 for maroon.

Example 1

A box contains 3 green apples and 5 red apples. An apple is picked from the box and not replaced then a second apple is picked. Expressing the answer as a fraction in its simplest form, calculate

    (i)   the probability that both apples picked are green.

    (ii)   the probability that the two apples picked are of different colours.

Solution

i.   P(both green) = P(1st G) X P(2nd G)

$$= \frac{3}{8} \text{ X } \frac{2}{7} = \frac{3}{28}$$

ii.   P(dif. colours) = P(1st G & 2nd R) + P(1st R & 2nd G)

$$= \left(\frac{3}{8} \text{ x } \frac{5}{7}\right) + \left(\frac{5}{8} \text{ x } \frac{3}{7}\right)$$

$$= \frac{15}{56} + \frac{15}{56}$$

$$= \frac{15}{28}$$

Example 2

A pack of eleven identical cards, are labelled 1 to 11. The cards re shuffled and placed upside down. If a card is picked at random from the pack, what is the probability that it is a prime numbered card?

Solution

The prime numbered cards are 2, 3, 5, 7, 11, total 5 cards

Therefore, P(prime numbered card) = $\frac{5}{11}$

Task

Two pupils are to represent a school at a Human Rights Conference. if the two are chosen at random from a group of 8 girls and 6 boys, calculate the probability that the two pupils picked

    (i)   are both girls,

    (ii)   at least one is a boy

23. Commercial and Social Arithmetic

This is everyday mathematics.

Example 1

Mataya's digital camera stores images on a memory card of capacity 512 units. Given that two thirds of the memory card is used, calculate the number of unused units, giving your answer correct to 2 decimal places.

Solution
if $\frac{2}{3}$ of the capacity is used, then $\frac{1}{3}$ is unused. Therefore, unused capacity = $\frac{1}{3}$ x 512 units = 170.67 units

Example 2

To write an examination, a candidate is required to pay an entry fee of K38.50 and K25.00 for each subject.

    (i)   if Mary intends to write 4 subjects, how much is she expected to pay?
    (ii)   Given that Martin paid a total of K188.50, how many subjects did he write?

Solution

(i)   Total cost = entry fee + total cost of 4 subjects
                = K38.50 + 4(K25.00)
                = **K138.5**
(ii)   amount paid for subjects = total cost – entry fee
                       = K188.50 – K38.50
                       = K150.0
        number of subjects written = total cost of subjects ÷ cost per subject
                       = 150.00 ÷ 25
                       = 6

Task

   (a) The cost of baking a birthday cake is K48 000.
        (i)   Ireen has an order of 15 such cakes. How much did she spend on baking the cakes?
        (ii)   Given that each cake was sold at K57 600, find the percentage profit
   (b) The price of a shirt was K55 000 after an increase of 10%. What was the price of the shirt before the increment?